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December 1977

MUTUAL ADMITTANCE BETWEEN SLOTS ON A CYLINDER OR CONE

S. W. Lee

R. Mittra

ELECTROMAGNETICS LABORATORY
DEPARTMENT OF ELECTRICAL ENGINEERING
ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
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In the present contract, ~~we~~ studied the mutual admittance between two thin slots on a conducting cylinder or a cone by ray techniques. Our main contribution in the development of an approximate asymptotic Green's function for the surface function, expressed in terms of Fock functions, is simple to evaluate, and gives excellent numerical results when compared with known exact solutions and/or experimental data. This report contains a brief administrative summary plus three attachments which give the technical details.

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by

S. W. Lee
R. Mittra

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TABLE OF CONTENTS

I. INTRODUCTION

The contract N00019-77-C-0127 was awarded to the University of Illinois by the Naval Air Systems Command for "Mutual Admittance Between Slots on a Cylinder or Cone" for a one-year period, 16 November 1976 to 15 November 1977. Mr. J. Willis of AIR-310B is the contract monitor.

This is the final report for the contract, covering personnel (section II), technical results (section III and attachments), publications and presentations (section IV).

The attachments have their own pagination. The report is page numbered in lower left-hand and right-hand corners.

II. PERSONNEL

S. W. Lee, Professor of Electrical Engineering, University of Illinois, principal investigator

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C. L. Law, Computer Programmer, University of Illinois

III. TECHNICAL RESULTS

In the design of a slot array on a conformed surface, a most important parameter is the mutual admittance Y_{12} between two slots. In the present contract, we have studied the following problems about Y_{12}:

(a) When the conformal surface is a conducting cylinder, a GTD solution of Y_{12} has been successfully developed. It applies to cylinders with radius greater than one wavelength, and gives excellent numerical results (error is within 0.25 dB in magnitude and several degrees in phase). Details are given in Attachment A.

(b) For the slot array on a cylinder, a simple approximate solution of Y_{12} is derived. It is generally valid when the separation between the slots is greater than two wavelengths (Attachment B).

(c) The GTD solution described in (a) has been generalized, so that it now can be used to calculate Y_{12} between slots on a general convex conducting surface. In particular, it was applied to slots on a cone, and the numerical results of Y_{12} are in good agreement with the experimental results (Attachment C).

IV. PUBLICATIONS AND PRESENTATIONS

(1) S. W. Lee, S. Safavi-Naini, and R. Mittra, "Mutual Admittance Between Slots on a Cylinder," University of Illinois, Electromagnetics Lab. Tech. Rept. 77-8, Urbana, Illinois, March 1977.

(2) S. W. Lee and S. Safavi-Naini, "Simple Approximate Formula for Mutual Admittance Between Slots on a Cylinder," University of Illinois, Electromagnetics Lab. Tech. Rept. 77-13, Urbana, Illinois, July 1977.

(3) S. W. Lee and R. Mittra, "Mutual Admittance Between Slots on a Cylinder or a Cone," Final Report, Contract N00019-77-C-0127 (including the above two reports as attachments), December 1977.

(4) S. W. Lee and S. Safavi-Naini, "Approximate Asymptotic Solution of Surface Field Due to a Magnetic Dipole on a Cylinder," to appear in IEEE Trans. Antenna Propagat., 1978.

(5) S. W. Lee and S. Safavi-Naini, "An Unexpected Result About Surface Rays," Digest of International Symposium on Antennas and Propagat., pp. 52-55, Stanford, CA, 20-22 June 1977.

(6) S. W. Lee, S. Safavi-Naini, and R. Mittra, "Mutual Admittance Between Slots on a Cylinder," paper presented at Program Review of Naval Air Systems Command, Hughes Aircraft Company, Culver City, CA, 29 March 1977.

ATTACHMENT A

Report 77-8

MUTUAL ADMITTANCE BETWEEN SLOTS ON A CYLINDER

ELECTROMAGNETICS LABORATORY
TECHNICAL REPORT NO. 77-8

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MUTUAL ADMITTANCE BETWEEN SLOTS ON A CYLINDER

S. W. Lee
S. Safavi-Naini
R. Mittra

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DEPARTMENT OF ELECTRICAL ENGINEERING
ENGINEERING EXPERIMENT STATION
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In the design of conformal slot array on the surface of a conducting cylinder, the calculation of the mutual admittance Y_{12} is a crucial step, which has been studied extensively in recent years. In this paper, we summarize, in a handbook format, all of the final formulas of Y_{12}, and present some typical numerical data.

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S. Safavi-Naini
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TABLE OF CONTENTS

LIST OF FIGURES

10

LIST OF TABLES

11

TABLE		Page
F-1	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 0$	55
F-2	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 0.5\lambda$	56
F-3	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 1\lambda$	57
F-4	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 2\lambda$	58
F-5	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 4\lambda$	59
F-6	UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 8\lambda$	60
F-7	COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS	61
F-8	COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS	62
F-9	COMPARISON OF ASYMPTOTIC SOLUTIONS	63

12

1. INTRODUCTION

In the design of a conformal slot array on the surface of a conducting cylinder, the calculation of the mutual admittance Y_{12} is a crucial step, which has been studied extensively in recent years. In this paper, we summarize, in a handbook format, all of the final formulas of Y_{12}, and present some typical numerical data.

2. STATEMENT OF PROBLEM

Referring to Figure 1, two identical slots, circumferential or axial, are located on the surface of an infinitely long cylinder. The geometrical parameters are

$$R = \text{radius of the cylinder} \tag{2.1}$$

$$(a,b) = \text{dimensions of the slot along } (\phi,z) \text{ directions (a is the arc length along the cylinder)} \tag{2.2}$$

$$(z_0,R\phi_0) = \text{center-to-center distances between slots} \tag{2.3}$$

$$s_0 = \sqrt{z_0^2 + (R\phi_0)^2} \tag{2.4}$$

$$\theta_0 = \tan^{-1}(z_0/R\phi_0) \tag{2.5}$$

The problem is to determine the mutual admittance between these two slots when kR is large.

13

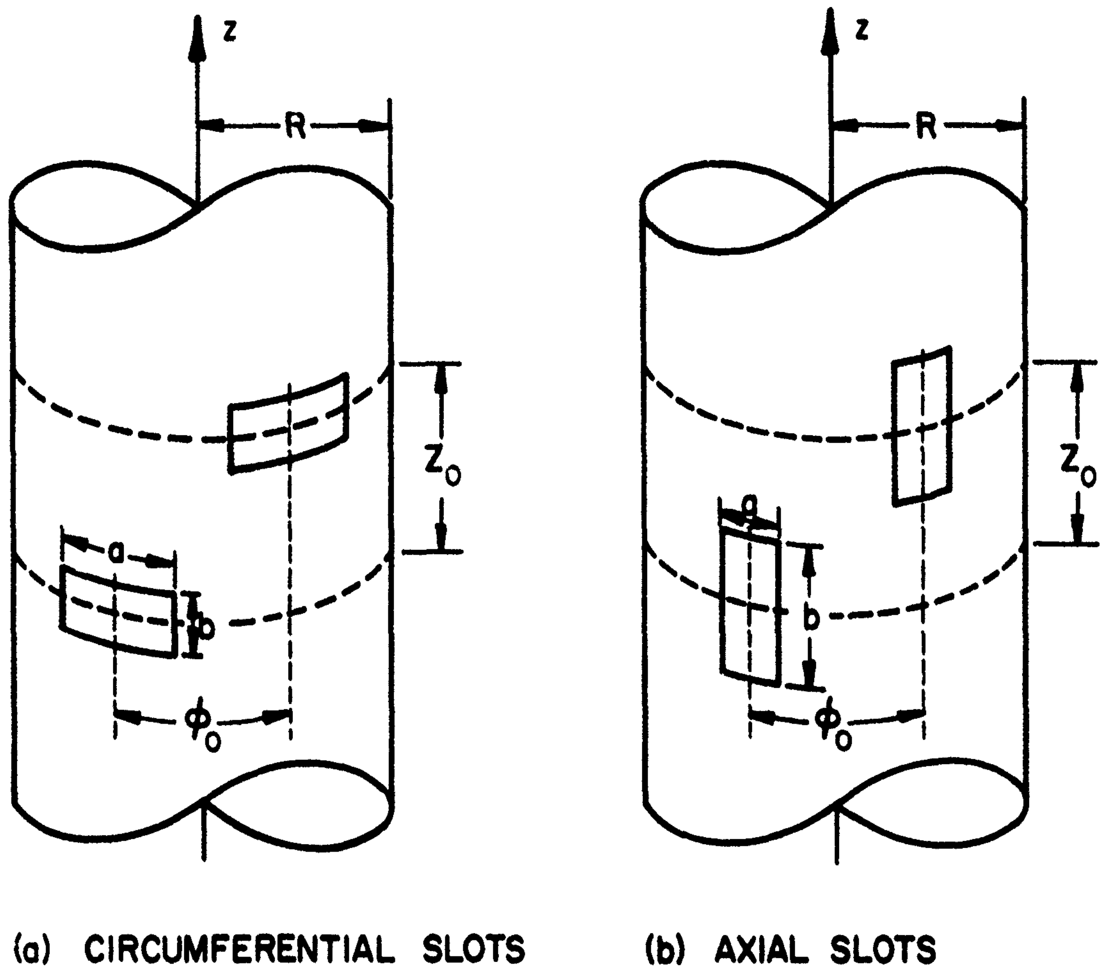

Figure 1. Two identical slots on the surface of a cylinder.

First let us define mutual admittance. Throughout this work we always assume that

(i) the slots are thin, and (2.6a)

(ii) their length is roughly a half-wavelength. (2.6b)

Then the aperture field in each slot can be adequately approximated by a simple cosine distribution, which is the so-called "one-mode" approximation. For example, if slot 1 is a circumferential (lower slot in Figure 1a), its aperture field under the "one-mode" approximation is given by (exp + $j\omega t$ time convention)

$$\vec{E} = V_1\vec{e}_1 \ , \ \vec{H} = I_1\vec{h}_1 \tag{2.7a}$$

where

$$\vec{e}_1 = \hat{z}\sqrt{\frac{2}{ab}} \cos\frac{\pi}{a} y \ , \ \vec{h}_1 = \hat{x} \times \vec{e}_1 \tag{2.7b}$$

$$y = R\phi \ . \tag{2.7c}$$

(V_1, I_1) are respectively the modal (voltage, current) of slot 1. The mutual admittance Y_{12} is defined by

$$Y_{12} = Y_{21} = \frac{I_{21}}{V_1} \tag{2.8}$$

where I_{21} is the induced current in slot 2 when slot 1 is excited by a voltage V_1 and slot 2 is short-circuited. An alternative expression for Y_{12} is

$$Y_{12} = \frac{1}{V_1 V_2} \iint_{A_2} \vec{E}_2 \times \vec{H}_1 \cdot d\vec{s}_2 \tag{2.9}$$

where

A_2 = aperture of slot 2

$\vec{H}_1$ = magnetic field when slot 1 is excited with voltage V_1, and slot 2 is covered by a perfect conductor

$\vec{E}_2$ = electric field when slot 2 is excited with voltage V_2, and slot 1 is covered by a perfect conductor.

Because $\vec{H}_1 = I_{21}\vec{h}_2$ and $\vec{E}_2 = V_2\vec{e}_2$, it is a simple matter to verify that (2.8) and (2.9) are equivalent [1].

There is an alternative definition of mutual admittance. Instead of (2.7), a modal voltage $\overline{V}_1$ (with a bar) may be defined through the expression for the aperture field of slot 1 as follows:

$$\vec{E} = \hat{z}\,\frac{1}{b}\,\overline{V}_1 \cos\frac{\pi}{a}\,y \tag{2.10a}$$

or equivalently

$$\overline{V}_1 = \int_{-b/2}^{b/2} (\hat{z}\cdot\vec{E})_{y=0}\,dz \quad . \tag{2.10b}$$

Then a different mutual admittance $\overline{Y}_{12}$ is defined by (2.9) after replacing (V_1,V_2) by $(\overline{V}_1,\overline{V}_2)$. It can be easily shown that

$$\overline{Y}_{12} = \frac{a}{2b}\,Y_{12} \quad . \tag{2.11}$$

Two remarks are in order: (i) In the limiting case that $b \to 0$, Y_{12} goes to zero as b , whereas $\overline{Y}_{12}$ approaches a constant independent of b. (ii) For the special case $a = \lambda/2$ and $R \to \infty$, it is $\overline{Y}_{12}$ that is identical to the mutual impedance Z_{12} between two corresponding dipoles calculated by the classical Carter's method [2], [3], [4]. (iii) When the slots are excited by waveguides (transmission lines), one often uses Y_{12} $(\overline{Y}_{12})$. From here on, we will concentrate on Y_{12} instead of $\overline{Y}_{12}$.

The mutual admittance defined in (2.8) and (2.9) includes the self admittance Y_{11} as a special case which occurs when two slots coincide. (All the formulas of Y_{12} given in this paper, except for the one in Section 4, can be used for calculating Y_{11} by setting $\phi_0 \to 0$ and $z_0 \to 0$.)

3. EXACT HUGHES (GSP) MODAL SOLUTION

Once the one-mode approximation in (2.7) is accepted, Y_{12} can be determined exactly in terms of cylindrical modal functions, as has been done by Stewart, Golden, and Pridmore-Brown [5], [6]. The final result reads:

Circumferential slots

$$Y_{12} = \int_{-\infty}^{\infty} dk_z \sum_{m=-\infty}^{\infty} \psi(m,k_z) G(m,k_z) e^{-j(m\phi_0 + k_z z_0)} \tag{3.1}$$

where

$$\psi(m,k_z) = \frac{ab}{8\pi^2 R} \frac{\sin^2(k_z b/2)}{(k_z b/2)^2} \cdot \left\{ \frac{\sin (m\phi_a + \pi/2)}{(m\phi_a + \pi/2)} + \frac{\sin (m\phi_a - \pi/2)}{(m\phi_a - \pi/2)} \right\}^2 \tag{3.2}$$

$\phi_a = (a/2R)$

$$G(m,k_z) = Y_0 \left[\frac{jk}{k_t} \frac{H_m^{(2)'}(k_t R)}{H_m^{(2)}(k_t R)} + \left(\frac{mk_z}{k_t^2 R} \right)^2 \frac{k_t}{jk} \frac{H_m^{(2)}(k_t R)}{H_m^{(2)'}(k_t R)} \right] . \tag{3.3}$$

$$k_t = \begin{cases} \sqrt{k^2 - k_z^2} & , \text{ if } k \geq k_z \\ -j\sqrt{k_z^2 - k^2} & , \text{ if } k \leq k_z \end{cases} \tag{3.4}$$

Axial slots

$$Y_{12} = \int_{-\infty}^{\infty} dk_z \sum_{m=-\infty}^{\infty} \phi(m,k_z)\ F(m,k_z) e^{-j(m\phi_0 + k_z z_0)} \tag{3.5}$$

where

$$\phi(m,k_z) = \frac{ab}{8R} \left(\frac{\sin(m\phi_a)}{(m\phi_a)} \cdot \frac{\cos(k_z b/2)}{(k_z b/2)^2 - (\pi/2)^2} \right)^2 \tag{3.6}$$

$$F(m,k_z) = Y_0 \frac{k_t}{jk} \frac{H_m^{(2)}(k_t R)}{H_m^{(2)'}(k_t R)} \tag{3.7}$$

This solution is suitable for numerical calculation if (i) $z_0 < b$ for circumferential slots, and $z_0 < a$ for axial slots, (ii) kR is less than 20, and (iii) the medium is slightly lossy so that k has a small (negative) imaginary part. Based on this solution, extensive numerical results have been reported by Hughes Aircraft Company at Culver City [7], [8], [9].

4. EXACT UI MODAL SOLUTION

Under the one-mode approximation, another exact modal solution is given in [10]. This solution is derived from the Hughes (SGP) solution in Section 3 by a deformation of integration contour and an application of the Duncan transform [11]. THe final result reads

<u>Circumferential slots</u>

$$Y_{12} = G + jB \tag{4.1a}$$

$$G = \int_0^k \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \cos k_z z_0 \; \psi(m,k_z) R(m,k_z) \, dk_z \tag{4.1b}$$

$$B = \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \left\{ -\int_0^k R(m,k_z)\psi(m,k_z) \sin k_z z_0 \, dk_z + \int_0^{\infty} R(m,j\eta)\psi(m,j\eta) e^{-\eta z_0} \, d\eta \right\} . \tag{4.1c}$$

where

$$R(m,k_z) = -\frac{2}{\pi k_t R} \cdot \frac{k}{k_t} \cdot \left[\frac{1}{M_m^2(k_t R)} + \left(\frac{m k_z}{k_t k R} \right)^2 \frac{1}{N_m^2(k_t R)} \right] \tag{4.2}$$

$$M_m^2(\chi) = J_m^2(\chi) + Y_m^2(\chi) \tag{4.3}$$

$$N_m^2(\chi) = J_m'^2(\chi) + Y_m'^2(\chi) \tag{4.4}$$

$$\varepsilon_m = \begin{cases} 2 , & m = 0 \\ 1 , & m \neq 0 \end{cases} . \tag{4.5}$$

$\psi(m,k_z)$ is defined in (3.2) and k_t in (3.4) (4.6)

Axial slots

$$Y_{12} = \frac{8Y_0}{\pi kR} \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \left(\int_0^k \Phi(m,k_z) e^{-jk_z z_0} \frac{dk_z}{N_m^2(k_t R)} + j \int_0^{\infty} \Phi(m,j\eta) e^{-\eta z_0} \frac{d\eta}{N_m^2\left(R\sqrt{\eta^2 + k^2}\right)} \right) \tag{4.7}$$

where $\Phi(m,k_z)$ is defined in (3.6)

This solution is valid only if $z_0 > b$ for circumferential slots and $z_0 > a$ for axial slots. It is suitable for numerical calculation if kR is less than 20.

5. ASYMPTOTIC SOLUTION

The two modal solutions given in Sections 3 and 4 are based on fields in the Fourier transform domain. An alternative calculation of Y_{12} involves the field in the spatial domain, namely,

Circumferential slots

$$Y_{12} = \frac{-2}{ab} \int_{A_1} dy_1 dz_1 \int_{A_2} dy_2 dz_2 \left[\cos \frac{\pi}{a} y_1\right]\left[\cos \frac{\pi}{a}(y_2 - R\phi_0)\right] g_\phi(s,\theta) \tag{5.1}$$

Axial slots

$$Y_{12} = \frac{-2}{ab} \int_{A_1} dy_1 dz_1 \int_{A_2} dy_2 dz_2 \left[\cos \frac{\pi}{b} z_1\right]\left[\cos \frac{\pi}{b}(z_2 - z_0)\right] g_z(s,\theta) \tag{5.2}$$

where (y_n, z_n) = a typical point in the aperture of slot n (n = 1 or 2). (5.3)

A_n = aperture of slot n (5.4)

$$s = \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{5.5}$$

$$\theta = \tan^{-1}[(z_2 - z_1)/(y_2 - y_1)] \tag{5.6}$$

Several versions of the Green's functions g_ϕ and g_z have been approximately determined under the condition that $kR >> 1$. They are listed as follows:

OSU Asymptotic solution [12] [13]

$$g_\phi \sim G(s)\ [v(\xi)\ \sin^2\theta + (\frac{j}{ks})u(\xi)\ \cos^2\theta] \tag{5.7}$$

$$g_z \sim G(s)\ [v(\xi)\ \cos^2\theta + (\frac{j}{ks})u(\xi)\ \sin^2\theta] \tag{5.8}$$

PINY Asymptotic solution [9] [14]

$$g_\phi \sim G(s)\left(v(\xi)[\sin^2\theta + \frac{j}{ks}(1 - 3\sin^2\theta)] + \frac{j}{ks}\sec^2\theta[u(\xi) - v_1(\xi)\sin^2\theta]\right) \tag{5.9}$$

$$g_z \sim G(s)\ v(\xi)\ [\cos^2\theta + \frac{j}{ks}(2 - 3\cos^2\theta)] \tag{5.10}$$

UI Asymptotic solution [15]

$$g_\phi \sim G(s)\left(v(\xi)[\sin^2\theta + \frac{j}{ks}\cos 2\theta] + (\frac{j}{ks})\ u(\xi)[\cos^2\theta(1 - \frac{2j}{ks}) + (\frac{j}{ks})\sin^2\theta] + j(\sqrt{2}\ kR/\cos^2\theta)^{-2/3}[v'(\xi)\ \sin^2\theta + (\tan^4\theta + \frac{j}{ks})u'(\xi)\ \cos^2\theta]\right) \tag{5.11}$$

$$g_z = G(s)\left(v(\xi)[\cos^2\theta - \frac{j}{ks}\cos 2\theta] + (\frac{j}{ks})u(\xi)[\sin^2\theta(1 - \frac{2j}{ks}) + (\frac{j}{ks})\cos^2\theta] + j(\sqrt{2}\ kR/\cos^2\theta)^{-2/3}[v'(\xi)\cos^2\theta + (1 + \frac{j}{ks})u'(\xi)\sin^2\theta]\right) \tag{5.12}$$

where

$$G(s) = \frac{k^2 Y_0}{2\pi j}\frac{e^{-jks}}{ks}, \quad Y_0 = (120\pi)^{-1} \tag{5.13}$$

$$\xi = (k\cos^4\theta/2R^2)^{1/3}s \tag{5.14}$$

The Fock functions, u, v, etc., can be calculated from the following two representations:

For $0 \le \xi \le 0.7$

$$v(\xi) \sim 1 - \frac{\sqrt{\pi}}{4} e^{j\pi/4}\xi^{3/2} + \frac{7j}{60}\xi^3 + \frac{7\sqrt{\pi}}{512} e^{-j\pi/4}\xi^{9/2} - 4.141 \times 10^{-3}\xi^6 \quad (5.15)$$

$$u(\xi) \sim 1 - \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} + \frac{5j}{12}\xi^3 + \frac{5\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} - 3.701 \times 10^{-2}\xi^6 \quad (5.16)$$

$$v_1(\xi) \sim 1 + \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} - \frac{7j}{12}\xi^3 - \frac{7\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} + 4.555 \times 10^{-2}\xi^6 \quad (5.17)$$

$$v'(\xi) \sim \frac{3\sqrt{\pi}}{8} e^{-j3\pi/4}\xi^{1/2} + \frac{7j}{20}\xi^2 + \frac{63\sqrt{\pi}}{1024} e^{-j\pi/4}\xi^{7/2} - 2.485 \times 10^{-2}\xi^5 \quad (5.18)$$

$$u'(\xi) \sim \frac{3}{4}\sqrt{\pi}\, e^{-j3\pi/4}\xi^{1/2} + \frac{5j}{4}\xi^2 + \frac{45\sqrt{\pi}}{128} e^{-j\pi/4}\xi^{7/2} - 2.221 \times 10^{-1}\xi^5 \quad (5.19)$$

<u>For $0.7 \le \xi \le \infty$</u>

$$v(\xi) \simeq e^{-j\pi/4}\sqrt{\pi}\,\xi^{1/2} \sum_{n=1}^{10} (t'_n)^{-1} e^{-j\xi t'_n} \quad (5.20)$$

$$u(\xi) \simeq e^{j\pi/4} 2\sqrt{\pi}\,\xi^{3/2} \sum_{n=1}^{10} e^{-j\xi t_n} \quad (5.21)$$

$$v_1(\xi) \simeq e^{j\pi/4} 2\sqrt{\pi}\,\xi^{3/2} \sum_{n=1}^{10} e^{-j\xi t'_n} \quad (5.22)$$

$$v'(\xi) \simeq \frac{1}{2} e^{-j\pi/4}\sqrt{\pi}\,\xi^{-1/2} \sum_{n=1}^{10} (1 - j2\xi t'_n)(t'_n)^{-1} e^{-j\xi t'_n} \quad (5.23)$$

$$u'(\xi) \simeq e^{j\pi/4} 3\sqrt{\pi}\,\xi^{1/2} \sum_{n=1}^{10} \left(1 - j\,\frac{2}{3}\,\xi t_n\right) e^{-j\xi t_n}\,. \quad (5.24)$$

where $t_n = |t_n|\exp(-j\pi/3)$, $t'_n = |t'_n|\exp(-j\pi/3)$, and

n	$\lvert t_n \rvert$	$\lvert t'_n \rvert$
1	2.33811	1.01879
2	4.08795	3.24820
3	5.52056	4.82010
4	6.78671	6.16331
5	7.99413	7.37218

n	$\lvert t_n \rvert$	$\lvert t'_n \rvert$
6	9.02265	8.48849
7	10.04017	9.53545
8	11.00852	10.52766
9	11.93602	11.47506
10	12.82878	12.38479

It has been verified through several hundred numerical examples that the UI asymptotic solution given above is in excellent agreement (within a quarter db in magnitude and a few degrees in phase) with the exact model solution for all slot separations (ϕ_0, z_0) provided that $kR \geq 5$.

In using the asymptotic solutions for calculating the self admittance Y_{11}, care must be exercised in avoiding the singularity in the Green's function which occurs at $s = 0$. A most convenient way to avoid this apparent difficulty is to (i) use a large number of points for the two surface integrals in (5.1) and (5.2), and (ii) shift slightly the integration nets for this two surface integrals.

6. EXACT PLANAR SOLUTION

In the limit $kR \to \infty$ the Green's function of the UI solution in (5.11) and (5.12) is reduced to

$$g_\phi = G(s)[\sin^2\theta + \frac{1}{ks}(2 - 3\sin^2\theta)(1 - \frac{1}{ks})] \tag{6.1}$$

$$g_z = G(s)[\cos^2\theta + \frac{1}{ks}(2 - 3\cos^2\theta)(1 - \frac{1}{ks})] \quad . \tag{6.2}$$

When (6.1) and (6.2) are used in (5.1) and (5.2), we obtain the exact solution (under the "one-mode" approximation of course) for two slots on an infinitely large, conducting plane.

7. APPROXIMATE SOLUTION

Based on the UI asymptotic solution, a simple approximate solution, is reported in [10], i.e.,

Circumferential slots

$$Y_{12} \simeq -\frac{8ab}{\pi^2}[S(b\sin\theta)\, C(a\sin\theta)]^2\, \bar{g}_\phi \quad . \tag{7.1}$$

Axial slots

$$Y_{12} \simeq - \frac{8ab}{\pi^2} [S(a \cos \theta)\ C(b \sin \theta)]^2\ \bar{g}_z \tag{7.2}$$

where

$$S(x) = \frac{\sin (kx/2)}{(kx/2)} \quad , \quad C(x) = \frac{\cos (kx/2)}{1 - (kx/\pi)^2} . \tag{7.3}$$

The (simplified) Green's functions $\bar{g}_\phi$ and $\bar{g}_z$ are given by

$$\bar{g}_\phi = G(s)\left[v(\xi)\left(\sin^2 \theta + \frac{j}{ks} \cos 2\theta\right) + \frac{j}{ks}\, u(\xi) \cos^2 \theta \right.$$
$$\left. + ju'(\xi)(\sqrt{2}\ kR \cos \theta)^{-2/3} \sin^4 \theta \right] \tag{7.4}$$

$$\bar{g}_z = G(s)\left[v(\xi)\left(\cos^2 \theta - \frac{j}{ks} \cos 2\theta\right) + \frac{j}{ks}\, u(\xi) \sin^2 \theta\right]. \tag{7.5}$$

This solution gives an accurate numerical result (within several percent in magnitude and less than 5^0 in phase) provided that $kR \geq 10$ and the slot separation is greater than two wavelengths.

8. CONCLUDING REMARKS

Based on extensive numerical data, we conclude that Y_{12} (including Y_{11} as a special case) can be best calculated by

(i) Hughes modal solution if $kR \leq 5$ and z_0 is less than the axial dimension of the slot,

(ii) UI modal solution if $kR \leq 5$ and z_0 is greater than the axial dimension of the slot, and

(iii) UI asymptotic solution if $kR \geq 5$ for all slot separations.

If several percents of error are acceptable, the approximate solution can be used if $kR \geq 10$ and the slot separation is greater than two wavelengths.

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APPENDIX A: NUMERICAL RESULTS

By using the formulas of Y_{12} presented in the text, we have analyzed the following 6 slots:

Slot	Type	Dimension	Suggested by
A	Circumf.	0.9" x 0.4" (f = 9 GHz)	Aerospace Hughes
B	Circumf.	0.5λ x 0.01λ (R in inch)	Hansen
C	Axial	0.4" x 0.9" (f = 9 GHz)	Aerospace Hughes
D	Circumf.	0.5λ x 0.01λ (R in λ)	Hansen
E	Circumf.	0.5λ x 0.2λ	Hansen
F	Axial	0.5λ x 0.2λ	

In all tables, Y_{12} is listed in (db, phase in degree) format where db = 20 $\log_{10}$ (Y_{12} in mho). In all figures, the normalized phase of Y_{12} is equal to $\mathrm{Arg}(Y_{12} \exp jks_0)$.

DATA SET A OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two circumferential slots on an infinitely long cylinder is calculated from the

* (Exact) Hughes modal solution
* (Exact) UI modal solution
* UI asymptotic solution
* OSU asymptotic solution
* PINY asymptotic solution.

The parameters are

* Frequency: $f = 9$ GHz, $k = 4.7878$ (inch)$^{-1}$, $\lambda = 1.3123''$
* Cylinder: $R = 1.991''$ unless specified otherwise
* Slot A: Circumferential

  $a = 0.9'' = 0.6858\lambda$

  $b = 0.4'' = 0.3048\lambda$

  $|Y_{11}| = 1.70747 \times 10^{-3}$ mho $= -55.35$ db

  $Y_g = 1.8155 \times 10^{-3}$ mho
* Center-to-center distance between two slots is $(R\phi_0, z_0)$.

(2) Y_{12} is listed in (db value, phase in degree), where

db value = $20 \log_{10}$ ($|Y_{12}|$ in mho).

(3) Data are presented in

TABLE A-1: $\phi_0 = 0$ and various z_0

A-2: $z_0 = 2''$ and various ϕ_0

A-3: $z_0 = 0$ and various ϕ_0

A-4: $\phi_0 = 0$ and various z_0.

Figure A-1: Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

A-2 Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

A-3: $|Y_{12}|$ on a cylinder (UI modal solution) and that on a plane as a function of z_0.

A-4: Y_{12} on a cylinder as a function of the radius R of the cylinder.

TABLE A-1

Y_{12} OF SLOT A FOR $\phi_o = 0$

	Modal		Asymptotic			Exact Planar
z_o	Hughes	UI	UI	OSU	PINY	R=∞
0.5"	-62.62 db -72^o	-62.62 -72^o	-62.54 -72^o	-64.22 -43^o	-61.7 -68^o	-63.69 -67^o
2"	-71.87 -117^o	-71.78 -117^o	-71.66 -116^o	-73.67 -100^o	-70.96 -118^o	-73.53 106^o
8"	-82.3 33^o	-81.84 34^o	-81.83 37^o	-85.46 55^o	-80.80 34^o	-85.4 54^o
16"		-86.48 -4^o	-86.6 -1^o	-91.41 20^o	-85.26 -4^o	-91.40 19^o
40"		-91.95 -115^o	-92.46 -110^o	-99.34 -83^o	-90.83 -112^o	-99.33 -83^o

TABLE A-2

Y_{12} OF SLOT A FOR z_o = 2"

	Modal		Asymptotic		
ϕ_o	Hughes	UI	UI	OSU	PINY
0°	-71.87 db -117°	-71.78 -117°	-71.66 -116°	-73.67 -100°	-70.96 -118°
30°	-77.60 175°	-77.42 175°	-77.69 177°	-79.25 170°	-76.6 172°
60°	-89.98 -4°	-90.00 -3°	-90.17 -1°	-91.11 6°	-88.41 -10°
90°	-103.15 116°	-102.52 120°	-103.10 116°	-103.83 119°	-101.69 106°

TABLE A-3

Y_{12} OF SLOT A FOR z_o = 0

ϕ_o	Modal	Asymptotic		
	Hughes	UI	OSU	PINY
30°	-81.33 db -77°	-81.34 -75°	-89.72 -62°	-83.14 -60°
40°	-89.87 168°	-90.02 170°	-98.66 174°	-91.11 -180°
50°	-96.37 58°	-96.72 61°	-105.95 58°	-97.43 69°
60°	-101.97 -49°	-102.48 -47°	-112.59 -55°	-102.93 -39°

TABLE A-4

UI SOLUTIONS OF Y_{12} OF SLOT A FOR $\phi_o = 0$

z_o	Modal	Asymptotic	z_o	Modal	Asymptotic
0.5"	-62.62 db -72°	-62.54 -72°	11"	-84.06 -70°	-84.06 -68°
1"	-66.82 155°	-66.71 155°	12"	-84.61 15°	-84.65 18°
2"	-71.78 -117°	-71.66 -116°	13"	-85.12 100°	-85.20 103°
3"	-74.78 -31°	-74.67 -30°	14"	-85.63 -175°	-85.70 -172°
4"	-76.89 54°	-76.89 54°	15"	-86.09 -90°	-86.17 -86°
5"	-78.51 139°	-78.44 141°	16"	-86.48 -4°	-86.60 -1°
6"	-79.85 -136	-79.77 -134	17"	-86.85 81	-87.01 84
7"	-80.94 -51°	-80.88 -49°	18"	-87.24 166°	-87.38 170°
8"	-81.84 34°	-81.83 37°	20"	-87.91 -24°	-88.08 -19°
9"	-82.65 119°	-82.66 122°	30"	-90.33 110°	-90.68 115°
10"	-83.40 -156°	-83.40 -153°	40"	-91.95 -115°	-92.46 -110°

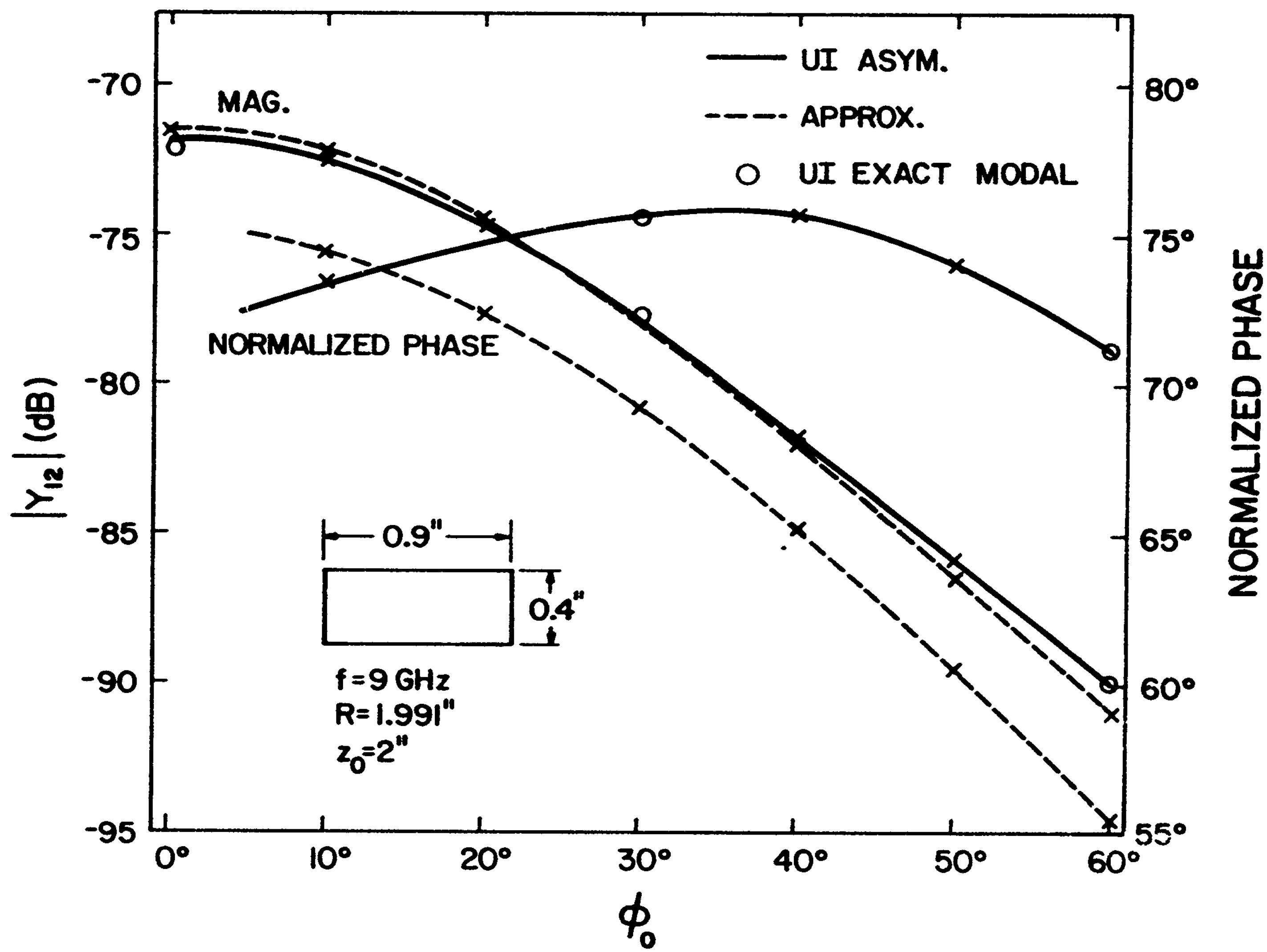

Figure A-1. Mutual admittance Y_{12} between two circumferential slots as a function ϕ_0.

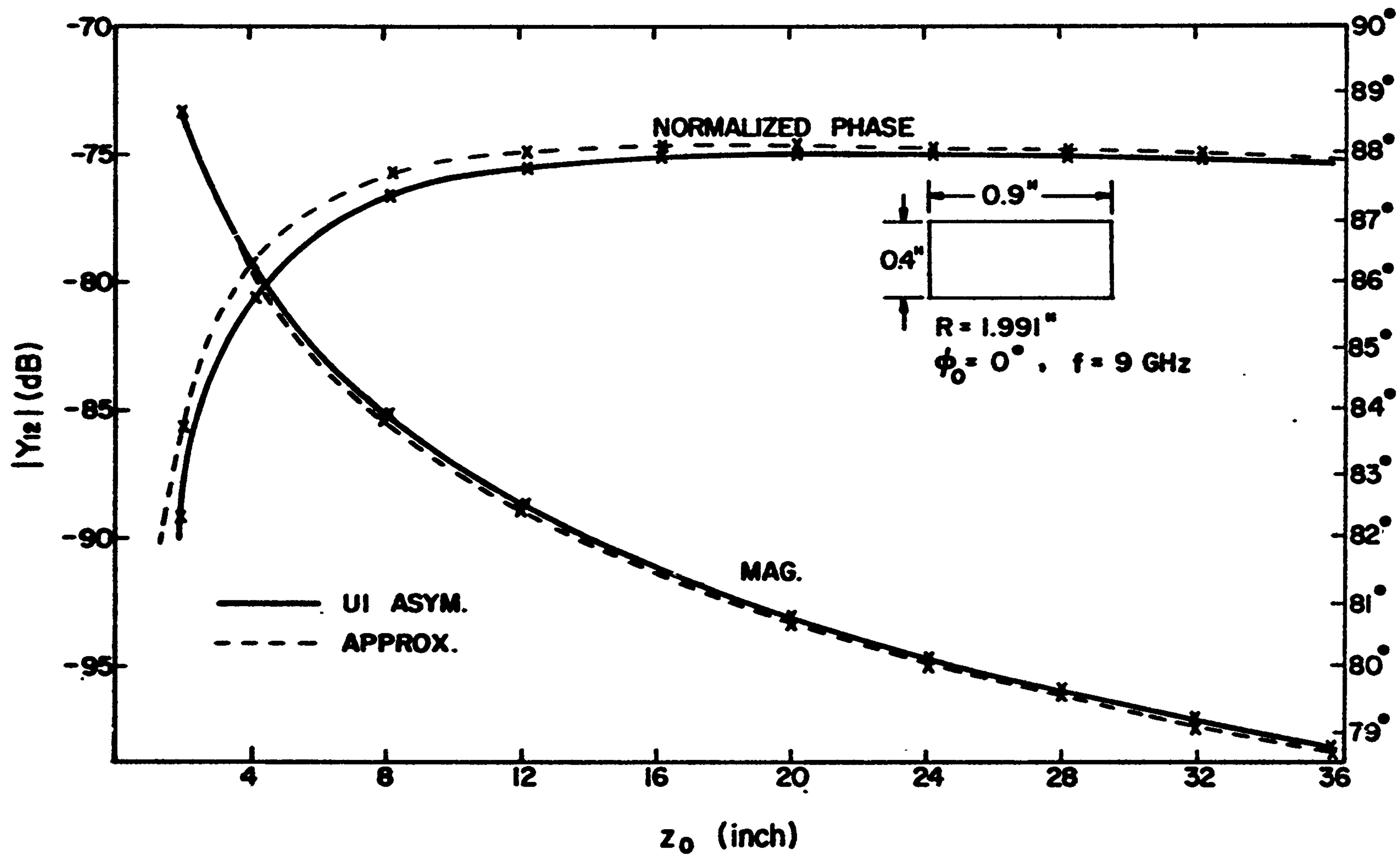

Figure A-2. Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

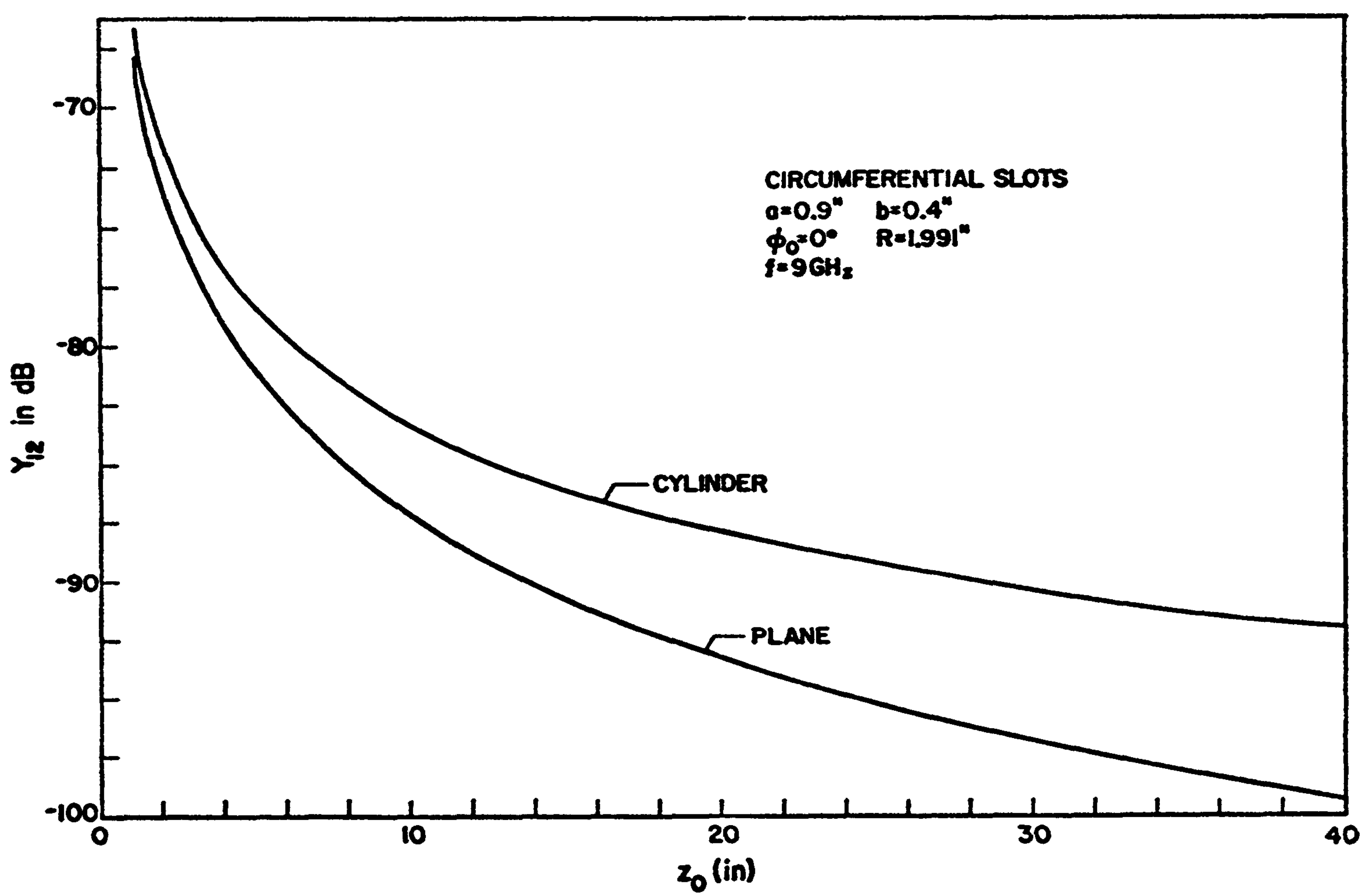

Figure A-3: $|Y_{12}|$ on a cylinder (UI modal solution) and that on a plane as a function of z_0.

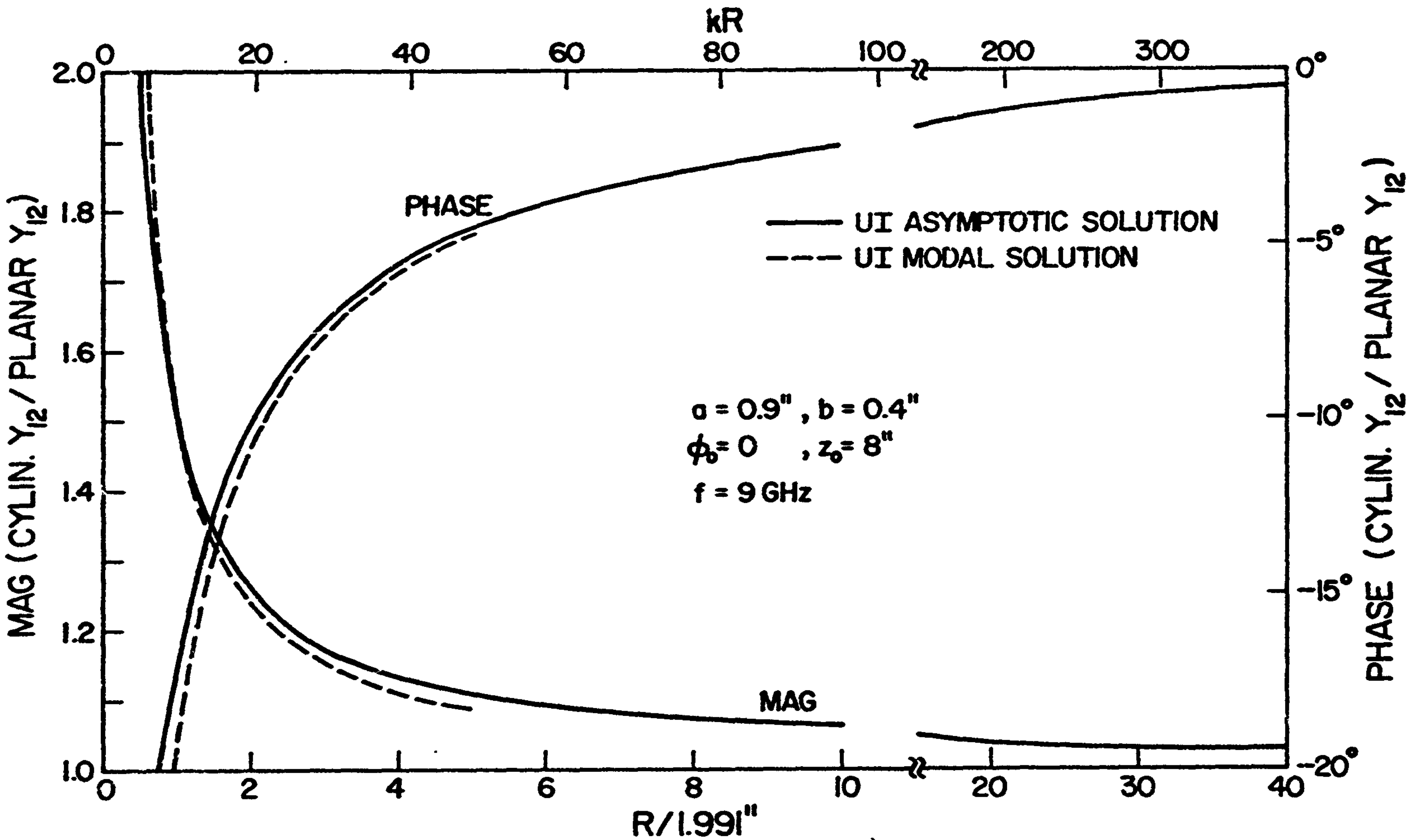

Figure A-4: Y_{12} on a cylinder as a function of the radius R of the cylinder. Y_{12} is normalized by Y_{12} on a plane which is 5.37×10^{-5} exp(j53.55°) mho.

DATA SET B OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two circumferential slots on an infinitely long cylinder is calculated from the

* (Exact) UI modal solution
* UI asymptotic solution.

The parameters are

* Frequency: $f = 9$ GHz, $k = 4.787787\ (\text{inch})^{-1}$, $\lambda = 1.3123''$
* Cylinder: $R = 1.991''$, $3.777''$, $6''$
* Slot B: Circumferential

  $a = 0.656168'' = 0.50\lambda$

  $b = 0.013123'' = 0.01\lambda$

* Center-to-center distance between two slots is $(R\phi_0, z_0)$.

(2) Y_{12} is listed in (db value, phase in degree), where

db value = $20 \log_{10} (|Y_{12}|$ in mho).

(3) Data are presented in

TABLE B-1: $\phi_0 = 0$ and various z_0

B-2: $z_0 = 2''$ and various ϕ_0

B-3: $z_0 = 8''$ and various ϕ_0.

B-4: Comparison of Hughes and UI solutions

TABLE B-1

UI SOLUTIONS OF Y_{12} OF SLOT B FOR $\phi_o = 0$

	R = 1.991"		R = 3.777"		R = 6"		Exact Planar R=∞
z_o	Modal	Asymp	Modal	Asymp	Modal	Asymp	
0.5"	-92.00 db -79^o	-92.03 -78^o	-92.48 -77^o	-92.52 -78^o	-92.70 -76^o	-92.74 -76^o	-93.11 -74^o
1"	-96.31 152^o	-96.28 153^o	-96.97 156^o	-96.92 159^o	-97.24 157^o	-97.19 157^o	-97.61 155^o
2"	-101.33 -117^o	-101.32 -116^o	-102.20 -113^o	-102.17 -113^o	-102.56 -111^o	-102.54 -111^o	-103.20 -109^o
4"	-106.50 54^o	-106.51 56^o	-107.70 60^o	-107.66 61^o	-108.23 63^o	-108.77 63^o	-109.10 67^o
8"	-111.48 36^o	-111.56 37^o	-113.13 42^o	-113.11 43^o	-113.85 46^o	-113.81 46^o	-115.08 53^o
16"	-116.13 -4^o	-116.35 -1^o	-118.37 5^o	-118.38 6^o	-119.36 10^o	-119.33 10^o	-121.10 20^o

TABLE B-2

UI SOLUTIONS OF Y_{12} OF SLOT B FOR $z_o = 2"$

	R = 1.991"		R = 3.777"		R = 6.0"	
ϕ_o	Modal	Asymp	Modal	Asymp	Modal	Asymp
10^o	-102.01 db -125^o	-102.04 -125^o	-104.18 -140^o	-104.22 -140^o	-106.89 -177^o	-106.94 -177^o
20^o	-103.94 -149^o	-104.11 -148^o	-109.18 142^o	-109.36 143^o	-115.80 11^o	-115.93 12^o
30^o	-106.86 172^o	-107.20 173^o	-115.53 27^o	-115.75 28^o	-124.77 140^o	-124.95 141^o
45^o	-112.51 92^o	-112.98 93^o	-125.07 169^o	-125.40 170^o	-136.67 106^o	-136.82 105^o
60^o	-119.01 -11^o	-119.28 -9^o	-134.48 -81^o	-134.38 -77^o	-148.07 51^o	-147.24 44^o
90^o	-131.40 110^o	-131.83 106^o	-148.22 132^o	-150.57 113^o	-155.92 -170^o	-165.47 -102^o

TABLE B-3

UI SOLUTIONS OF Y_{12} OF SLOT B FOR z_o = 8"

	R = 1.991"		R = 3.777"		R = 6.0"	
ϕ_o	Modal	Asymp	Modal	Asymp	Modal	Asymp
10°	-111.63 db 32°	-111.74 34°	-113.45 34°	-113.47 34°	-114.44 26°	-114.46 26°
20°	-112.08 24°	-112.29 26°	-114.40 9°	-114.54 9°	-116.18 -34°	-116.32 -34°
30°	-112.83 11°	-113.18 13°	-115.94 -32°	-116.26 -32°	-118.94 -130°	-119.21 -129°
45°	-114.41 -17°	-115.12 -16°	-119.29 -122°	-119.82 -121°	-124.43 27°	-124.85 29°
60°	-116.70 -56°	-117.70 -55°	-123.69 118°	-124.22 121°	-131.31 127°	-131.37 130°
90°	-122.98 -161°	-124.10 159°	-134.62 169°	-134.27 172°	-146.21 -132°	-145.33 146°

TABLE B-4

COMPARISON OF HUGHES AND UI SOLUTIONS

		R = 1.991"			R = 3.777"			R = 6"		
ϕ_o	z_o	Hughes Modal	UI		Hughes Modal	UI		Hughes Modal	UI	
			Modal	Asymp		Modal	Asymp		Modal	Asymp
0°	0.5"	-92.3 db -79°	-92 -79°	-92.03 -78°	-92.83 -77°	-92.48 -77°	-92.52 -78°	-92.87 -76°	-92.70 -76°	-92.74 -76°
	1"	-96.5 153°	-96.31 152°	-96.28 153°	-97.18 157°	-96.97 156°	-96.92 159°	-97.34 156°	-97.24 157°	-97.19 157°
	8"	-112.02 33°	-111.5 36°	-111.56 37°	-113.65 40°	-113.13 42°	-113.11 43°	-114.42 44°	-113.85 46°	-113.81 46°
	16"	-117.08 -6°	-116.13 -4°	-116.35 -1°	-119.27 3°	-118.37 5°	-118.38 6°		-119.36 10°	-119.33 10°
45°	2"	-112.73 91°	-112.51 92°	-112.98 93°	-125.43 168°	-125.07 169°	-125.40 170°	-137.17 104°	-136.7 106°	-136.82 105°

DATA SET C OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two <u>axial</u> slots on an infinitely long cylinder is calculated from the

* (Exact) UI modal solution
* UI asymptotic solution

The parameters are

* Frequency: $f = 9$ GHz, $k = 4.7877\ (\text{inch})^{-1}$, $\lambda = 1.3123''$
* Cylinder: $R = 1.991''$, and other values
* Slot C: Axial

  $a = 0.4'' = 0.3048\lambda$

  $b = 0.9'' = 0.6858\lambda$

* Center-to-center distance between two slots is $(R\phi_0, z_0)$.

(2) Y_{12} is listed in (db value, phase in degree), where db value = $20 \log_{10}$ ($|Y_{12}|$ in mho)

(3) Data are presented in

TABLE C-1: $\phi_0 = 0$, $R = 1.991''$, and various z_0.

C-2: $z_0 = 1.5''$, $R = 1.991''$, and various ϕ_0.

C-3: $\phi_0 = 0$, $z_0 = 8''$, and various R.

Figure C-1: $|Y_{12}|$ on a cylinder (UI modal solution) and that on a plane as a function of z_0.

TABLE C-1

Y_{12} OF SLOT C FOR $\phi_o = 0^\circ$

z_o	Modal	Asymp	z_o	Modal	Asymp
1"	-77.38 db -59°	-77.28 -59°	12"	-123.86 134°	-123.55 130°
2"	-92.00 8°	-91.86 6°	14"	-127.50 -51°	-126.23 -59°
3"	-99.48 89°	-99.25 86°	16"	-128.96 115°	-128.55 112°
4"	-104.68 172°	-104.36 170°	18"	-131.64 -68°	=130.60 -76°
5"	-108.88 -103°	-108.28 -106°	20"	-133.39 102°	-132.43 95°
6"	-111.94 -17°	-111.48 -21°	24"	-136.07 81°	-135.59 77°
7"	-114.61 68°	-114.17 64°	28"	-138.79 72°	-138.27 60°
8"	-116.93 151°	-116.5 149°	32"	-141.24 59°	-140.59 42°
9"	-119.28 -122°	-118.55 -126°	36"	-143.68 39°	-142.63 25°

TABLE C-2

Y_{12} OF SLOT C FOR z_0 = 1.5"

ϕ_0	Modal	Asymptotic
0^o	-86.58 db 151^o	-86.31 149^o
30^o	-86.41 -26^o	-85.15 -38^o
60^o	-87.43 84^o	-85.77 72^o
90^o	-93.02 169^o	-91.04 156^o

TABLE C-3

Y_{12} OF SLOT C FOR $\phi_o = 0$ and $z_o = 8''$

R	Modal	Asymptotic
0.995"	-118.07 db 150^o	-116.55 148^o
1.991"	-116.93 151^o	-116.50 149^o
3.982"	-116.91 150^o	-116.47 149^o
5.973"	-116.90 154^o	-116.46 149^o
7.964"	-116.89 154^o	-116.45 149^o
11.946"	-116.84 153^o	-116.45 149^o
15.928"	-116.82 153^o	-116.45 149^o
19.910"		-116.44 149^o

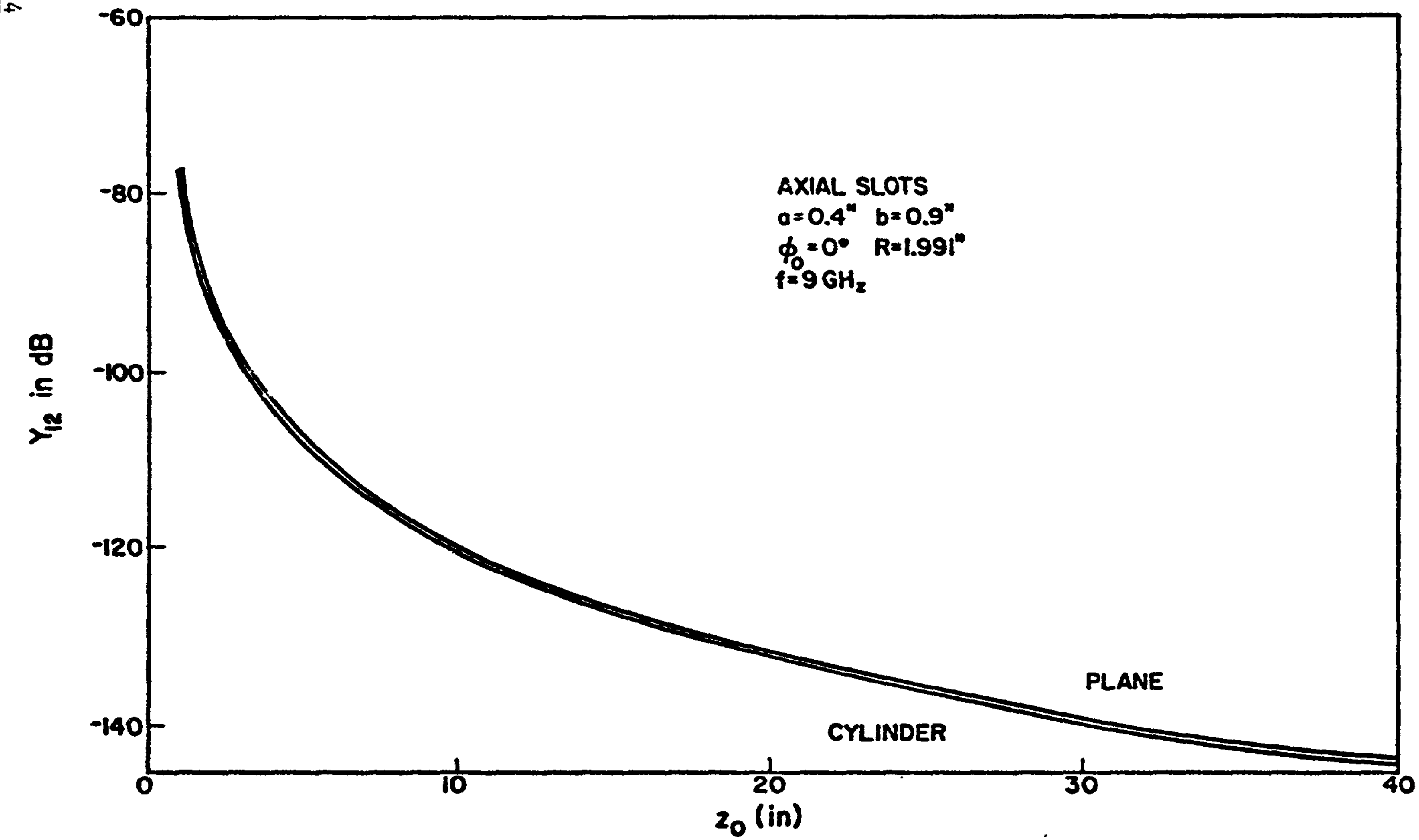

Figure C-1: $|Y_{12}|$ on a cylinder (UI modal solution) and that on a plane as a function of z_0.

DATA SET D OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two circumferential slots on an infinitely long cylinder from the

* (Exact) UI modal solution
* UI asymtotic solution

The parameters are

* Cylinder: $R = 1\lambda, 2\lambda, 4\lambda, 10\lambda, \infty$ (planar)
* Slot D: Circumferential

  $a = 0.5\lambda$

  $b = 0.01\lambda$

* Center-to-center distance between two slots is $(R\phi_o, z_o)$

(2) Y_{12} is listed in (db value, phase in degree), where

db value = $20 \log_{10} (|Y_{12}|$ in mho)

(3) Data are presented in

TABLE D-1: $\phi_o = 0$, $R = 2\lambda$ and various z_o

D-2: $\phi_o = 0$ and various R and z_o

D-3: $\phi_o = 0$ and various R and z_o

D-4: $z_o = 0$ and various R and ϕ_o

D-5: $z_o = 1\lambda$ and various R and ϕ_o

D-6: $z_o = 5\lambda$ and various R and ϕ_o

TABLE D-1

UI SOLUTIONS OF Y_{12} OF SLOT D FOR $\phi_o = 0$ and $R = 2\lambda$

z_o	Modal	Asymptotic
1λ	-98.60 db 71°	-98.56 71°
2λ	-103.87 74°	-103.84 75°
3λ	-106.98 75°	-106.96 75°
4λ	-109.17 74°	-109.16 75°
5λ	-110.84 73°	-110.85 75°
6λ	-112.19 73°	-112.21 74°
7λ	-113.32 72°	-113.35 74°
8λ	-114.28 72°	-114.33 73°
9λ	-115.12 71°	-115.18 73°
10λ		-115.94 72°

TABLE D-2

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT D FOR $\phi_o = 0$

z_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$	Planar ($R = \infty$)
0	81.51 db 90°	81.51 90°	81.51 90°	81.51 90°	
1λ	-97.49 67°	-98.56 71°	-99.15 74°	-99.51 76°	-99.76 77°
2λ	-102.39 69°	-103.84 75°	-104.63 79°	-105.13 81°	-105.47 83°
3λ	-105.26 69°	-106.96 75°	-107.92 80°	-108.52 83°	-108.93 86°
4λ	-107.25 68°	-109.16 75°	-110.25 80°	-110.94 84°	-111.40 87°
5λ	-108.76 67°	-110.85 75°	-112.05 80°	-112.81 84°	-113.33 87°
6λ	-109.97 67°	-112.21 74°	-113.51 80°	-114.34 84°	-114.91 88°
7λ	-110.98 66°	-113.35 74°	-114.74 80°	-115.63 84°	-116.25 88°
8λ	-111.85 65°	-114.33 73°	-115.80 79°	-116.75 84°	-117.40 88°
9λ	-112.60 65°	-115.18 73°	-116.72 79°	-117.73 84°	-118.43 89°
10λ	-113.27 64°	-115.94 72°	-117.55 79°	-118.61 84°	-119.34 89°

TABLE D-3

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT D FOR $\phi_o = 0$

z_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ	Planar (R = ∞)
0.5λ	-93.01 db -119°	-93.83 -116°	-94.27 -115°	-94.55 -114°	-94.74 -113°
1.5λ	-100.34 -111°	-101.62 -106°	-102.32 -103°	-102.76 -100°	-103.05 -99°
2.5λ	-103.98 -111°	-105.56 -104°	-106.44 -100°	-106.99 -98°	-107.37 -95°

TABLE D-4

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT D FOR $z_o = 0$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
10^o	-7.62 db 90^o	-43.02 90^o	-106.09 -59^o	-124.64 -92^o
20^o	-43.02 90^o	-107.10 -62^o	-121.97 29^o	-140.01 -20^o
30^o	-98.90 19^o	-117.45 155.34^o	-131.85 122^o	-150.99 52^o
45^o	-112.59 -106^o	-128.38 -52^o	-143.53 84^o	-164.60 165^o
60^o	-121.31 143^o	-137.31 102^o		-175.95 -74^o

TABLE D-5

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT D FOR $z_o = 1\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
10^o	-98.12 db 62^o	-100.56 54^o	-105.46 9^o	-120.20 107^o
20^o	-99.93 48^o	-105.62 4^o	-116.60 -160^o	-136.65 -115^o
30^o	-102.69 26^o	-111.94 -75^o	-126.33 -18^o	-148.10 -17^o
45^o	-107.99 -24^o	-121.35 134^o	-138.27 -21^o	-162.12 111^o
60^o	113.88 -89^o	-129.91 -40^o	-148.52 -40^o	-174.13 -123^o

50

TABLE D-6

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT D FOR $z_o = 5\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
10^o	-108.90 db 66^o	-111.13 70^o	-112.73 61^o	-115.49 -25^o
20^o	-109.31 61^o	-111.98 54^o	-114.67 6^o	-121.98 39^o
30^o	-109.98 53^o	-113.34 29^o	-117.66 -83^o	-129.87 -24^o
45^o	-111.47 37^o	-116.23 -26^o	-123.42 91^o	-141.86 -72^o
60^o	-113.50 14^o	-119.88 -99^o	-130.02 -145^o	-153.21 159^o

DATA SET E OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two circumferential slots on an infinitely long cylinder is calculated from the

* UI asymptotic solution

The parameters are

*Cylinder: $R = 1\lambda,\ 2\lambda,\ 4\lambda,\ 10\lambda$

*Slot E: Circumferential

$a = 0.5\lambda$

$b = 0.2\lambda$

*Center-to-center distance between two slots is $(R\phi_o, z_o)$

(2) Y_{12} is listed in (db value, phase in degree), where

db value = $20 \log_{10} (|Y_{12}|$ in mho)

(3) Data are presented in

TABLE E-1: $z_o = o$, various ϕ_o and R

E-2: $z_o = 0.5\lambda$, various ϕ_o and R

E-3: $z_o = 1\lambda$, various ϕ_o and R

E-4: $z_o = 2\lambda$, various ϕ_o and R

E-5: $z_o = 4\lambda$, various ϕ_o and R

E-6: $z_o = 8\lambda$, various ϕ_o and R

E-7: Comparison of UI asymptotic and UI modal solutions

E-8: Comparison of UI asymptotic and UI modal solutions

Figure E-1: Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

E-2: Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

E-3: Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

E-4: Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

E-5: Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

TABLE E-1

UI ASYMPTOTIC SOLUTIONS OF Y_{12} OF SLOT E FOR $z_o = 0$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
30^o	-73.94 7^o	-91.47 153^o	-105.83 121^o	-124.96 52^o
45^o	-86.67 -110^o	-102.35 -54^o	-117.50 83^o	-138.57 165^o
60^o	-95.31 140^o	-111.28 101^o	-127.44 49^o	-149.93 77^o

TABLE E-2

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 0.5\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-67.67 db -117^o	-68.46 -114^o	-68.89 -112^o	-69.16 -111^o
10^o	-69.00 -122^o	-72.97 -132^o	-81.72 170^o	-98.38 -146^o
20^o	-72.67 -137^o	-82.21 164^o	-95.39 -39^o	-113.59 -49^o
30^o	-77.77 -165^o	-90.67 64^o	-105.02 75^o	-124.52 32^o
45^o	-85.89 130^o	-100.98 -116^o	-116.60 50^o	-138.17 150^o
60^o	-93.37 47^o	-109.75 51^o	-126.60 20^o	-149.69 -90^o

TABLE E-3

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 1\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-72.28 db 68^o	-73.34 73^o	-73.92 76^o	-74.28 78^o
10^o	-72.91 64^o	-75.33 55^o	-80.15 9^o	-94.52 105^o
20^o	-74.71 49^o	-80.31 3^o	-91.02 -161^o	-110.75 -116^o
30^o	-77.44 26^o	-86.49 -76^o	-100.56 -20^o	-122.14 -18^o
45^o	-82.65 -24^o	-95.69 132^o	-112.38 -22^o	-136.13 111^o
60^o	-88.42 -90^o	-104.13 -42^o	-122.57 -41^o	-148.13 -124^o

TABLE E-4

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 2\lambda$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
0^o	-77.24 db 70^o	-78.67 76^o	-79.46 80^o	-79.96 82^o
10^o	-77.52 67^o	-79.44 65^o	-81.77 37^o	-89.55 -148^o
20^o	-78.37 57^o	-81.60 31^o	-87.38 -79^o	-103.31 76^o
30^o	-79.73 42^o	-84.81 -21^o	-94.18 112^o	-114.68 -144^o
45^o	-82.59 9^o	-90.76 -130^o	-104.37 162^o	-128.88 15^o
60^o	-86.17 -35^o	-97.24 94^o	-113.88 177^o	-141.23 153^o

TABLE E-5

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 4\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-82.10 db 68^o	-84.01 75^o	-85.10 81^o	-85.78 84^o
10^o	-82.26 67^o	-82.26 67^o	-85.97 57^o	-89.37 -49^o
20^o	-82.73 61^o	-82.73 61^o	-88.41 -10^o	-97.35 -37^o
30^o	-83.51 52^o	-83.51 52^o	-92.03 -116^o	-106.24 -149^o
45^o	-85.21 32^o	-85.21 32^o	-98.74 26^o	-118.99 -108^o
60^o	-87.48 5^o	-87.48 5^o	-106.08 117^o	-130.69 -58^o

TABLE E-6

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 8\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-86.70 db 66^o	-89.18 73^o	-90.65 80^o	-91.60 85^o
10^o	-86.81 64^o	-89.38 70^o	-91.06 67^o	-92.96 14^o
20^o	-87.12 61^o	-89.97 60^o	-92.26 31^o	-96.69 171^o
30^o	-87.63 56^o	-90.93 43^o	-94.18 -28^o	-101.98 -140^o
45^o	-88.77 44^o	-93.03 5^o	-98.14 -156^o	-111.31 -35^o
60^o	-90.35 27^o	-95.78 -45^o	-102.98 34^o	-121.14 -47^o

TABLE E-7

COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS

z_o	ϕ_o	R = 1λ		R = 2λ	
		Modal	Asym.	Modal	Asym
1λ	0^o	-72.54 db 67^o	-72.28 68^o	-73.64 73^o	-73.34 73^o
	10^o	-73.12 63^o	-79.91 64^o	-75.54 55^o	-75.33 55^o
	20^o	-74.78 48^o	-74.71 49^o	-80.33 3^o	-80.31 3^o
	30^o	-77.34 25^o	-77.44 26^o	-86.37 -77	-86.49 76^o
	45^o	-82.3 -26^o	-82.65 -24^o	-95.62 130^o	-95.69 132^o
	60^o	-88.05 -91^o	-88.42 -90^o	-103.77 -41^o	-104.13 -42^o

TABLE E-8

COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS

ϕ_o	z_o	R = 1λ		R = 2λ		Planar (Exact)
		Modal	Asym.	Modal	Asym.	
0^o	0.5λ	-67.87 db -117^o	-67.67 -117^o	-68.69 -114^o	-68.46 -114^o	-69.35 -110^o
	1λ	-72.54 67^o	-72.28 68^o	-73.64 73^o	-73.34 73^o	-74.52 79^o
	2λ	-77.46 68^o	-77.24 70^o	-78.98 75^o	-78.67 76^o	-80.29 84^o
	4λ	-82.22 66^o	-82.10 68^o	-84.3 75^o	-84.01 75^o	-86.25 87^o
	8λ	-86.65 62^o	-86.7 66^o	-89.41 72^o	-89.18 73^o	-92.25 89^o

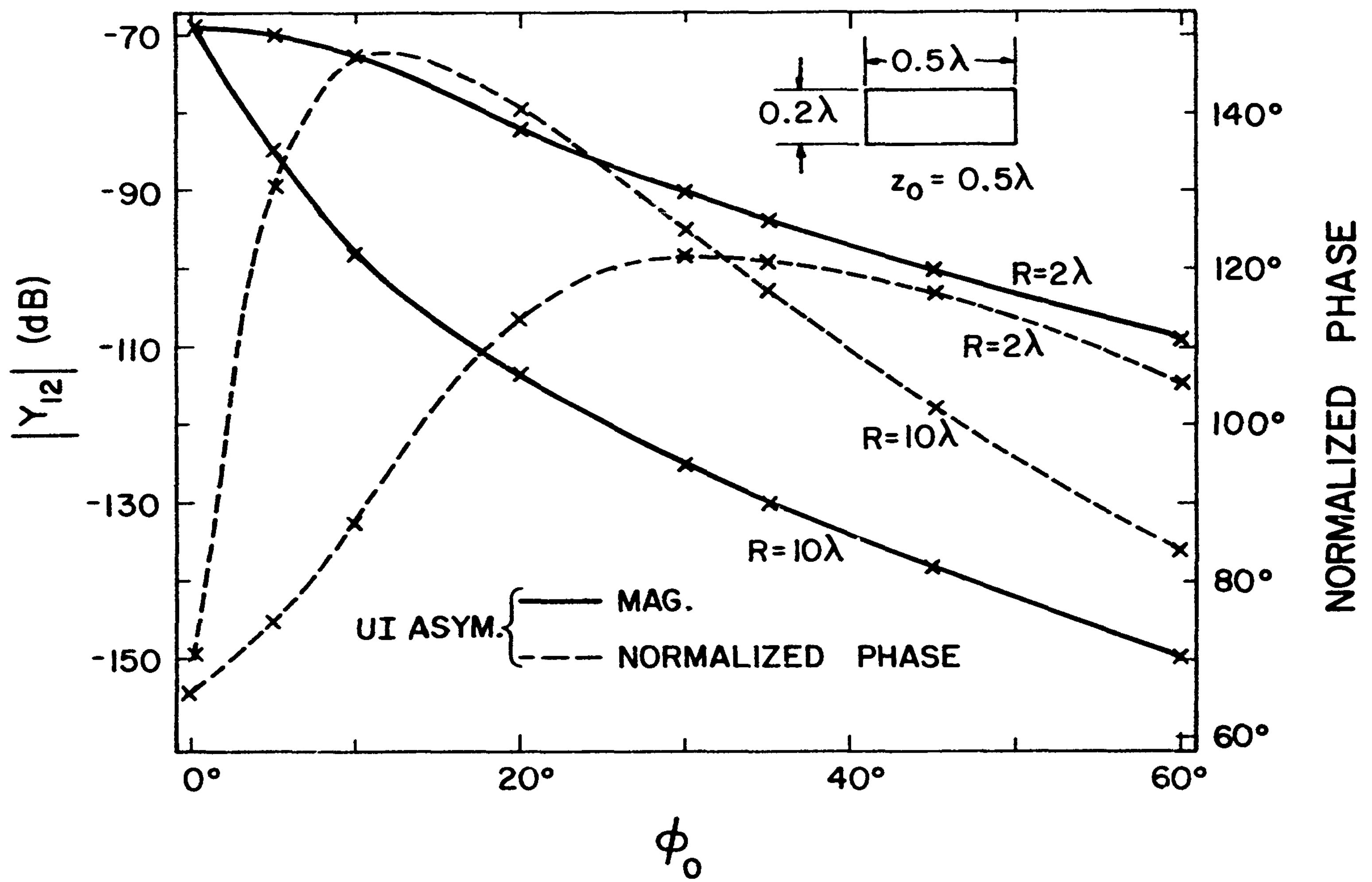

Figure E-1. Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

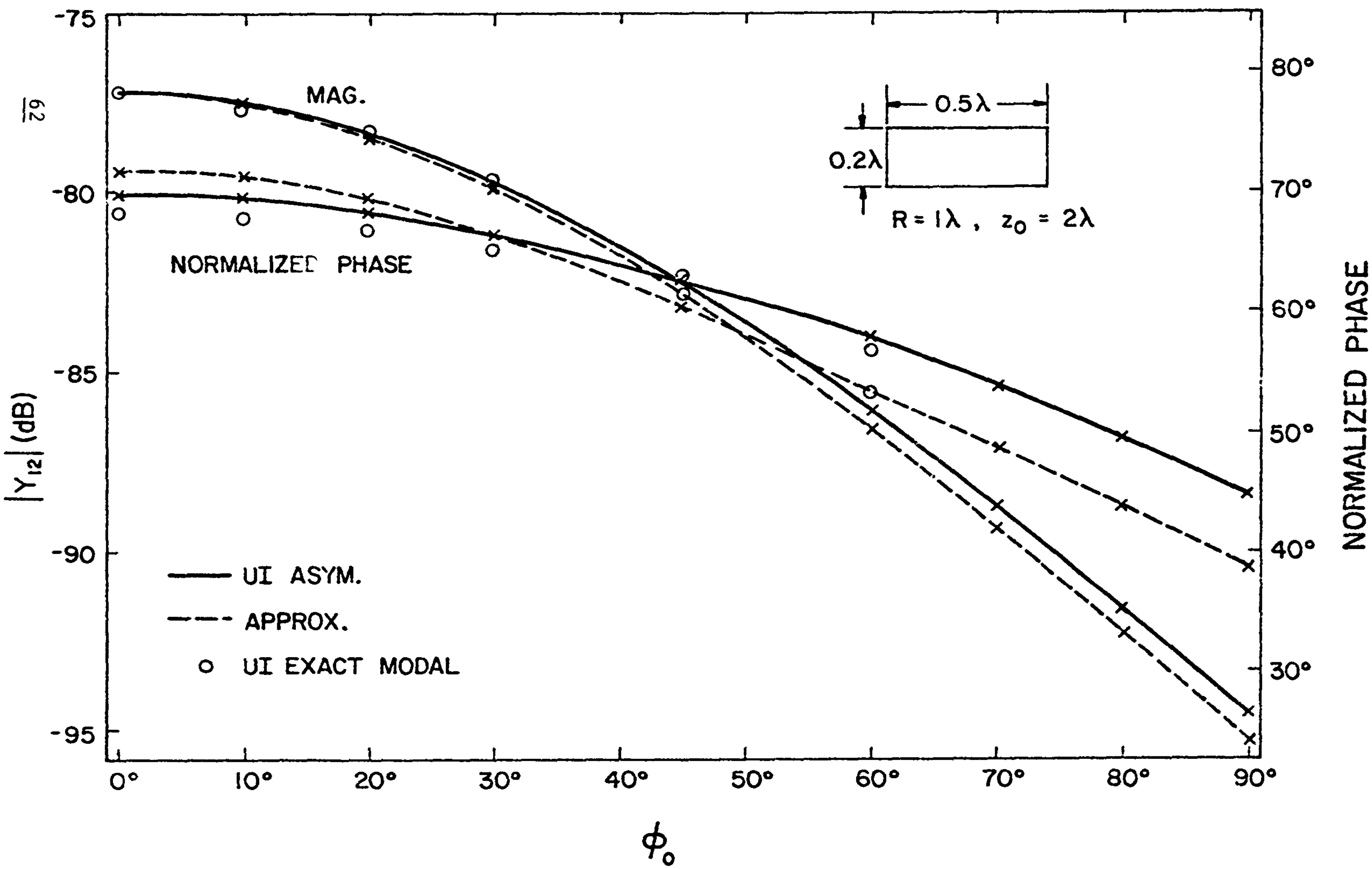

Figure E-2. Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

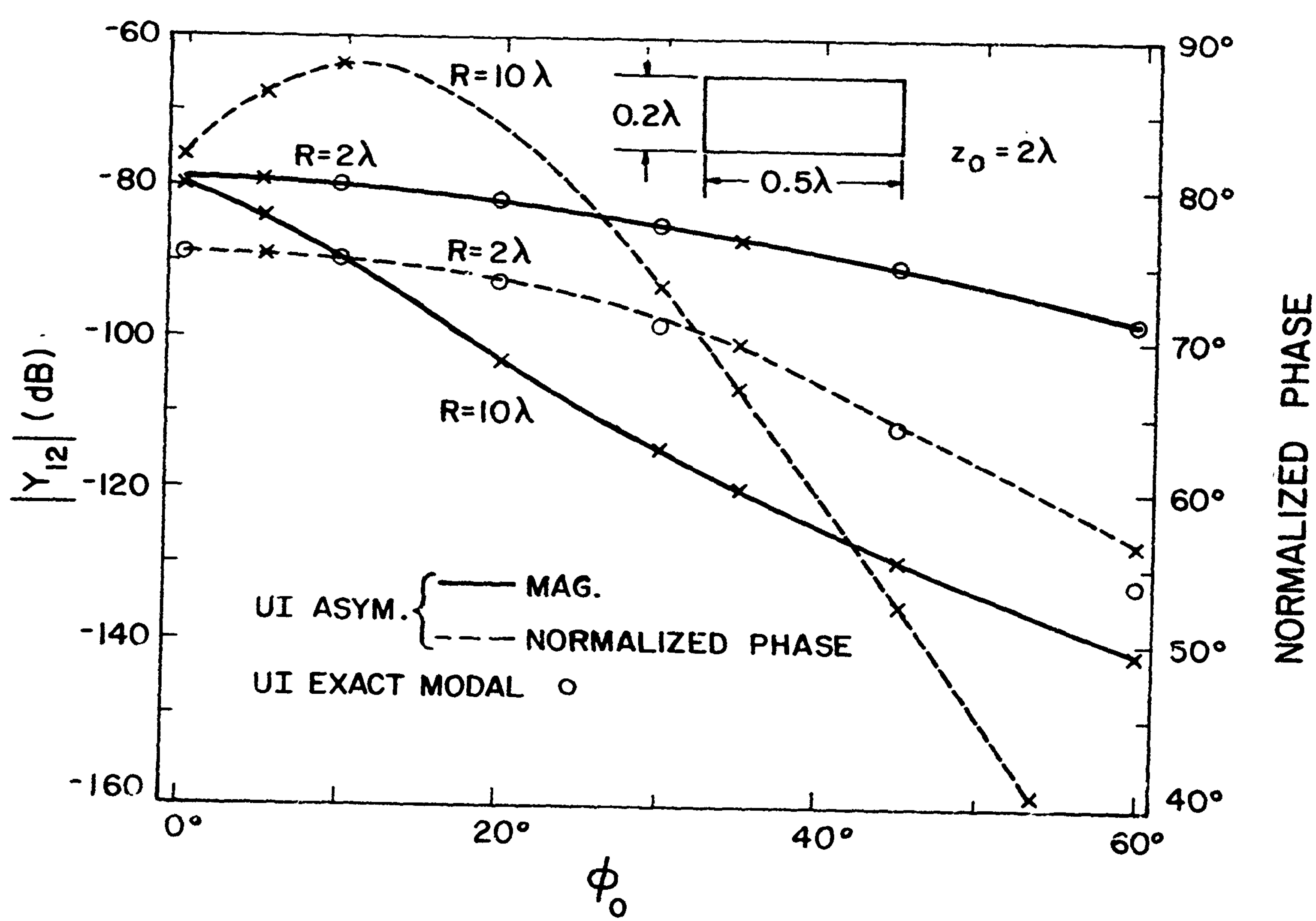

Figure E-3. Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

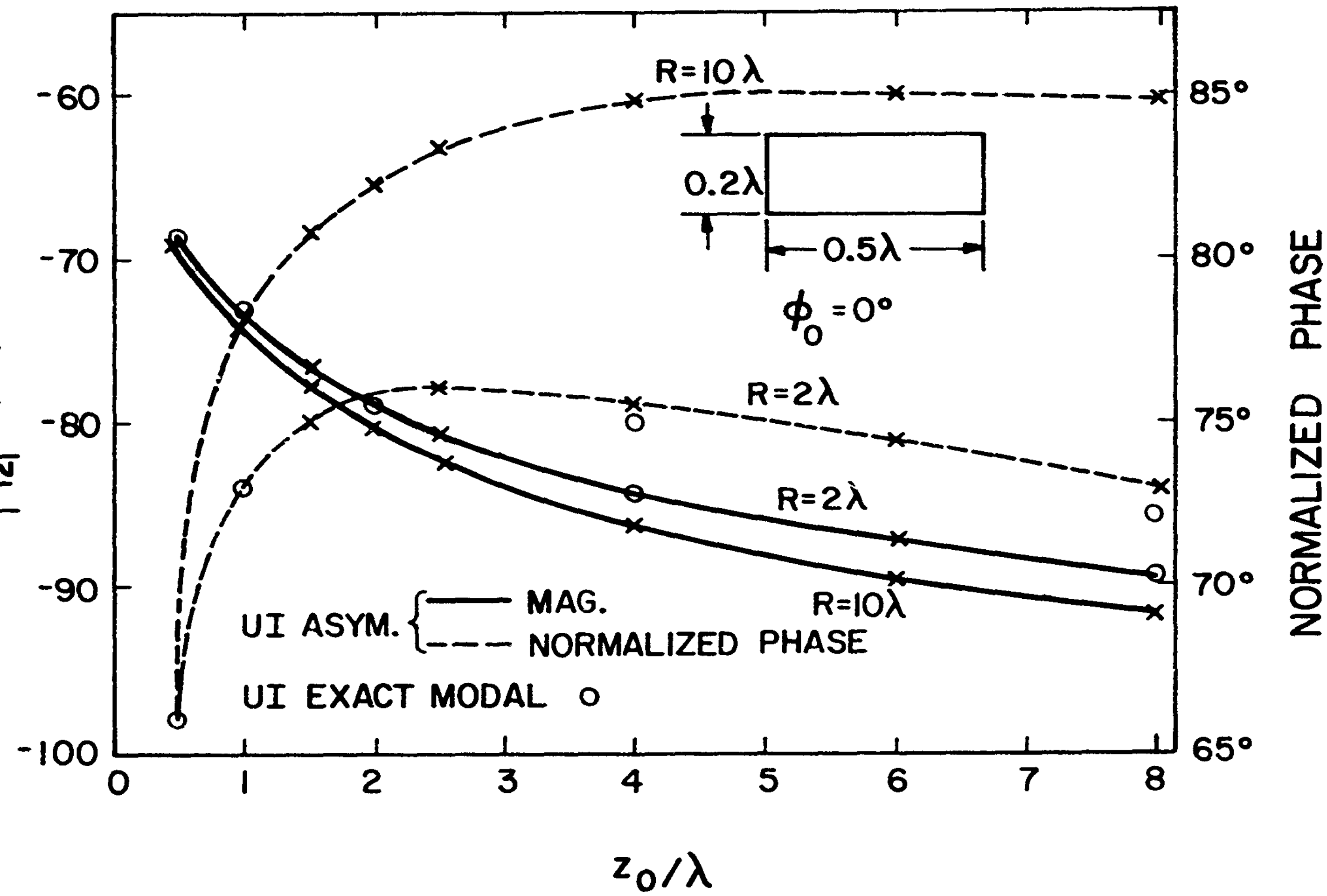

Figure E-4. Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

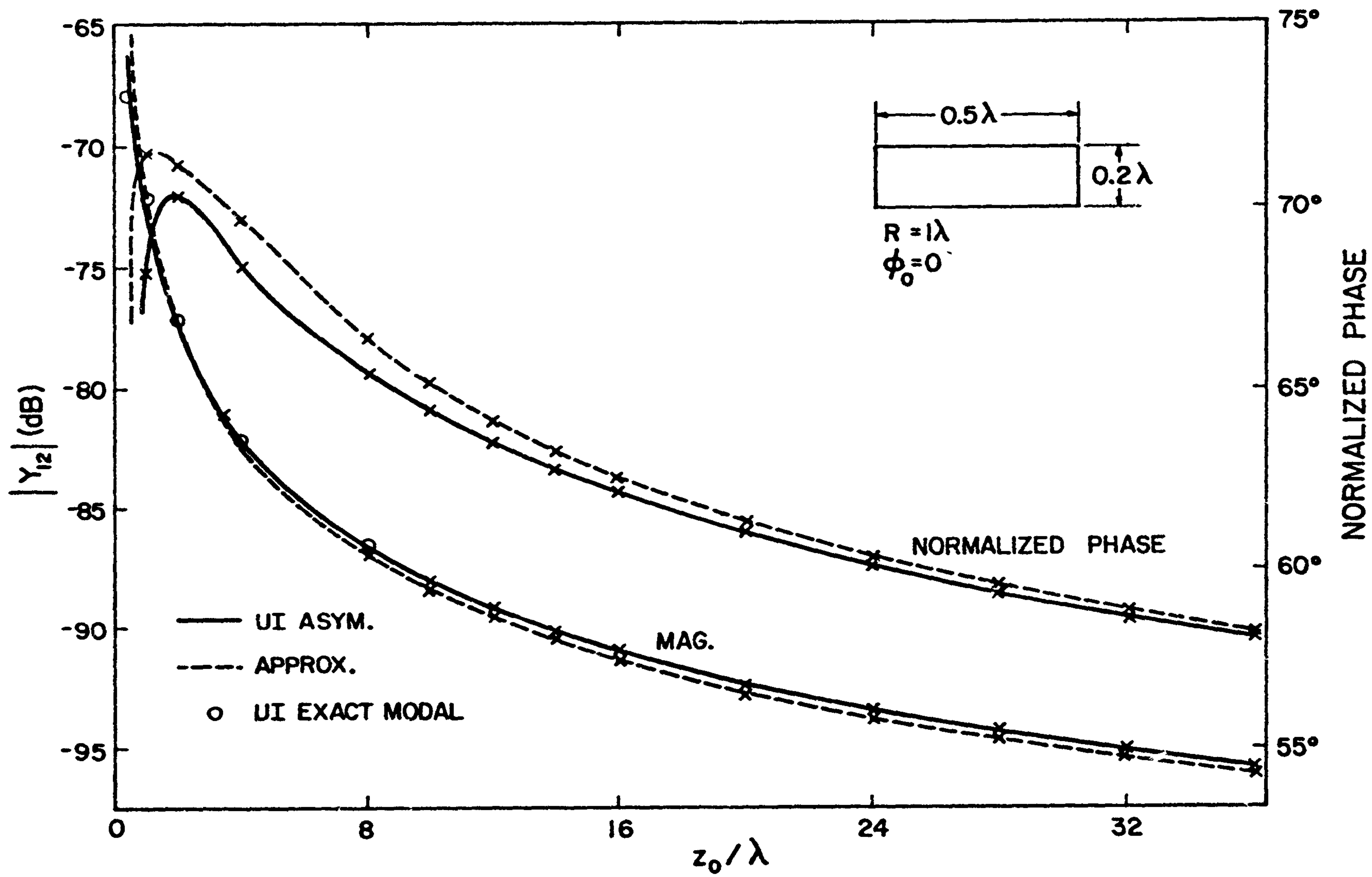

Figure E-5. Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

DATA SET F OF MUTUAL ADMITTANCE

(1) The mutual admittance Y_{12} between two axial slots on an infinitely long cylinder is calculated from the

*UI asymptotic solution

The parameters are

*Cylinder: $R = 1\lambda,\ 2\lambda,\ 4\lambda,\ 10\lambda$

*Slot F: Axial

$a = 0.2\lambda$

$b = 0.5\lambda$

*Center-to-center distance between two slots is $(R\phi_o, z_o)$

(2) Y_{12} is listed in (db value, phase in degree), where

db value = $20 \log_{10}$ ($|Y_{12}|$ in mho).

(3) Data are presented in

TABLE F-1: $z_o = 0$, various ϕ_o and R

F-2: $z_o = 0.5\lambda$, various ϕ_o and R

F-3: $z_o = 1\lambda$, various ϕ_o and R

F-4: $z_o = 2\lambda$, various ϕ_o and R

F-5: $z_o = 4\lambda$, various ϕ_o and R

F-6: $z_o = 8\lambda$, various ϕ_o and R

F-7: Comparison of UI asymptotic and UI modal solutions

F-8: Comparison of UI asymptotic and UI modal solutions

F-9: Comparison of asymptotic solutions

Figure F-1: Mutual admittance Y_{12} between two axial slots as a function of ϕ_0.

F-2: Mutual admittance Y_{12} between two axial slots as a function of ϕ_0.

F-3: Mutual admittance Y_{12} between two axial slots as a function of ϕ_0.

TABLE F-1

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 0$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
10^{o}	-63.59 -12^{o}	-67.11 -66^{o}	-72.13 178^{o}	-80.11 167^{o}
20^{o}	-67.13 -69^{o}	-72.57 173^{o}	-78.93 -75^{o}	-88.04 -110^{o}
30^{o}	-70.46 -131^{o}	-76.90 44^{o}	-83.98 26^{o}	-94.11 -32^{o}
45^{o}	-74.93 130^{o}	-82.36 -154^{o}	-90.41 -6^{o}	-102.17 82^{o}
60^{o}	-78.97 28^{o}	-87.25 6^{o}	-96.29 -39^{o}	-109.73 -164^{o}

TABLE F-2

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 0.5\lambda$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
0°	-70.14 db 25°	-70.11 25°	-70.10 25°	-70.09 26°
10°	-74.24 -20°	-76.61 -94°	-77.20 139°	-81.28 144°
20°	-76.84 -101°	-77.58 133°	-80.64 -102°	-88.34 -123°
30°	-77.48 -112°	-79.63 10°	-84.79 6°	-94.24 -41°
45°	-79.13 90°	-83.71 -179°	-90.78 -19°	-102.22 77°
60°	-81.68 -6°	-88.04 -14°	-96.49 -50°	-109.76 -168°

TABLE F-3

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 1\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0°	-86.65 db -173°	-86.63 -172°	-86.61 -172°	-86.60 -172°
10°	-87.35 172°	-87.92 134°	-86.18 33°	-84.30 78°
20°	-88.37 128°	-86.51 26°	-84.78 -180°	-89.22 -160°
30°	-88.07 70°	-85.58 -81°	-86.95 -51°	-94.63 -66°
45°	-87.12 -15°	-87.09 109°	-91.80 -60°	-102.39 60°
60°	-87.51 -99°	-90.13 -72°	-97.06 -81°	-109.84 179°

TABLE F-4

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 2\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-99.37 db -177^o	-99.34 -176^o	-99.33 -176^o	-99.33 -176^o
10^o	-99.72 176^o	-100.00 157^o	-98.96 93^o	-92.20 -144^o
20^o	-100.48 152^o	-99.39 85^o	-94.33 -67^o	-92.24 62^o
30^o	-100.83 115^o	-97.05 4^o	-93.13 109^o	-96.09 -163^o
45^o	-99.82 49^o	-95.40 -126^o	-95.21 150^o	-103.04 -7^o
60^o	-98.70 -15^o	-96.10 89^o	-99.12 161^o	-110.19 129^o

TABLE F-5

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 4\lambda$

ϕ_o	R = 1λ	R = 2λ	R = 4λ	R = 10λ
0^o	-111.56 db -178^o	-111.54 -178^o	-111.53 -178^o	-111.52 -178^o
10^o	-111.78 177^o	-111.97 168^o	-111.81 132^o	-105.41 -27^o
20^o	-112.38 164^o	-112.36 126^o	-108.23 21^o	-100.03 -35^o
30^o	-113.06 143^o	-111.16 67^o	-104.80 -104^o	-100.65 -154^o
45^o	-113.38 97^o	-108.48 -24^o	-103.49 28^o	-105.31 100^o
60^o	-112.64 47^o	-107.29 -123^o	-104.94 114^o	-111.46 -67^o

TABLE F-6

UI ASYMPTOTIC SOLUTIONS OF Y_{12} FOR $z_o = 8\lambda$

ϕ_o	$R = 1\lambda$	$R = 2\lambda$	$R = 4\lambda$	$R = 10\lambda$
0^o	-123.63 db -179^o	-123.61 -179^o	-123.61 -179^o	-123.60 -179^o
10^o	-123.78 -178^o	-123.92 173^o	-124.05 155^o	-120.63 54^o
20^o	-124.23 171^o	-124.56 150^o	-122.90 83^o	-113.13 -178^o
30^o	-124.87 159^o	-124.73 113^o	-119.69 -2^o	-110.52 -137^o
45^o	-125.87 131^o	-123.26 46^o	-116.53 -145^o	-111.57 -36^o
60^o	-126.32 94^o	-121.57 -21^o	-115.88 38^o	-115.46 -50^o

TABLE F-7

COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS

z_o	ϕ_o	R = 1λ		R = 2λ	
		Modal	Asym.	Modal	Asym.
1λ	0^o	-87.06 db -171^o	-86.65 -173^o	-86.83 -172^o	-86.63 -172^o
	10^o	-87.69 176^o	-87.35 172^o	-88.23 139^o	-87.92 134^o
	20^o	-88.91 139^o	-88.37 128^o	-87.64 35^o	-86.51 26^o
	30^o	-89.40 85^o	-88.07 70^o	-87.01 -72^o	-85.77 -81^o
	45^o	-89.19 2^o	-87.32 -15^o	-88.67 119^o	-87.30 109^o
	60^o	-89.84 -83^o	-87.72 -99^o	-91.86 -61^o	-90.36 -72^o

TABLE F-8

COMPARISON OF UI ASYMPTOTIC AND UI MODAL SOLUTIONS

ϕ_o	z_o	R = 1λ		R = 2λ		Plan.
		Modal	Asym.	Modal	Asym.	(Exact)
0^o	0.5λ	--	-70.14 db 25^o	--	-70.11 25^o	-70.08 26^o
	1λ	-87.06 -171^o	-86.65 -173^o	-86.83 -172^o	-86.63 -172^o	-86.6 -172^o
	2λ	-99.97 -174^o	-99.37 -177^o	-99.61 -176^o	-99.34 -176^o	-99.32 -176^o
	4λ	-112.43 -175^o	-111.56 -178^o	-111.93 -177^o	-111.54 -178^o	-111.52 -178^o
	8λ	-124.33 -174^o	-123.63 -179^o	-124.12 -177^o	-123.61 -179^o	-123.60 -179^o

TABLE F-9

COMPARISON OF ASYMPTOTIC SOLUTIONS

z_0	ϕ_0	R = 2λ			R = 10λ		
		UI Asym.	PINY	OSU	UI Asym.	PINY	OSU
2λ	0°	-99.34 db -176°	-99.42 -172°	-105.44 -172°	-99.33 -176°	-99.41 -172°	-105.42 -172°
	10°	-100.00 157°	-99.93 164°	-105.37 152°	-92.2 -144°	-92.51 -142°	-92.53 -143°
	20°	-99.39 85°	-99.71 98°	-101.89 78°	-92.24 62°	-92.46 64°	-92.45 65°
	30°	-97.05 4°	-97.85 17°	-98.23 4°	-96.09 -163°	-96.30 -161°	-96.30 -160°
	45°	-95.40 -126°	-96.16 -115°	-96.09 -120°	-103.04 -7°	-103.25 -5°	-103.25 -4°
	60°	-96.10 89°	-96.68 97°	-96.6 96°	-110.19 129°	-110.41 131°	-110.41 131°

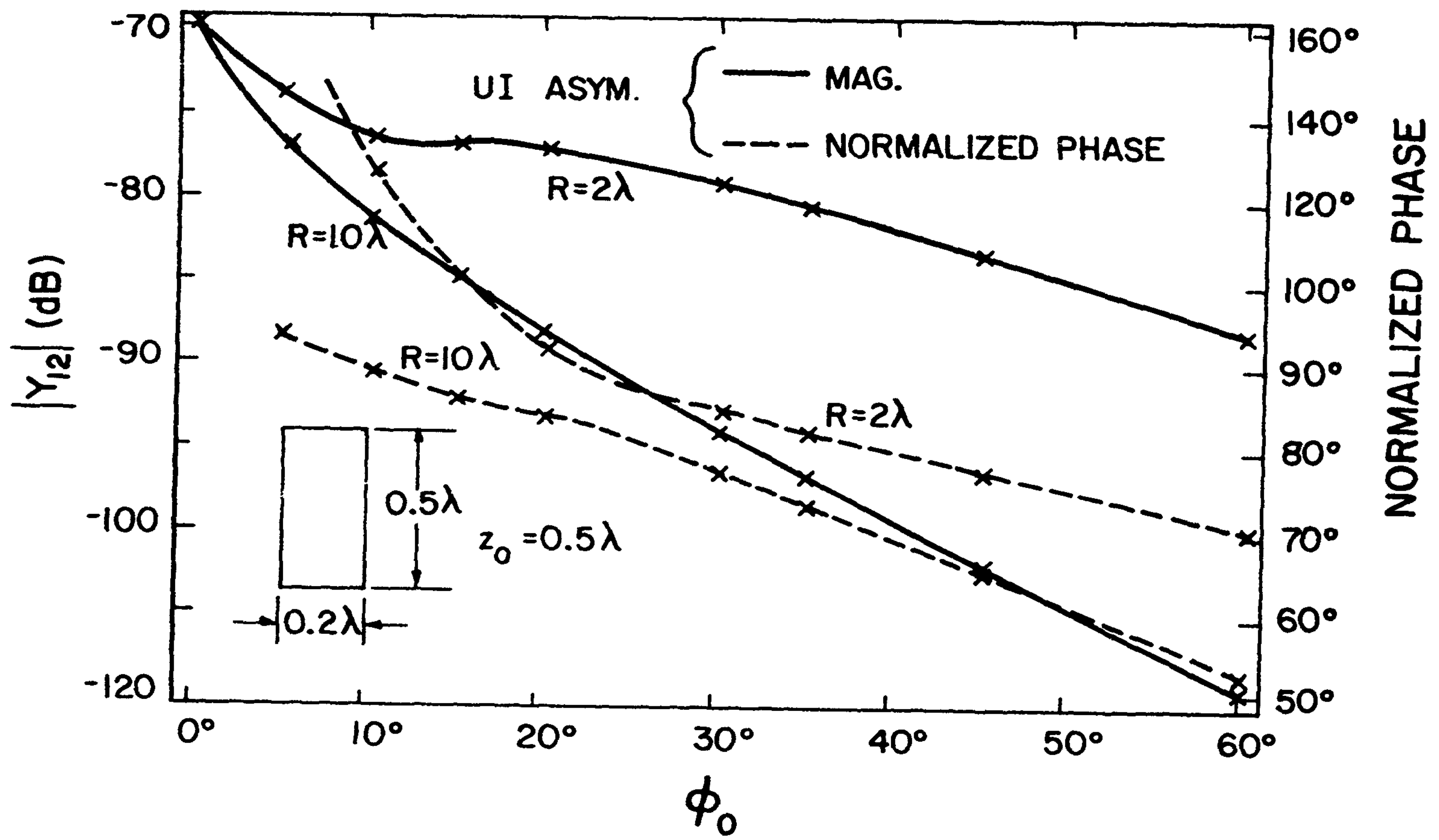

Figure F-1: Mutual admittance Y_{12} between two axial slots as a function of ϕ_0.

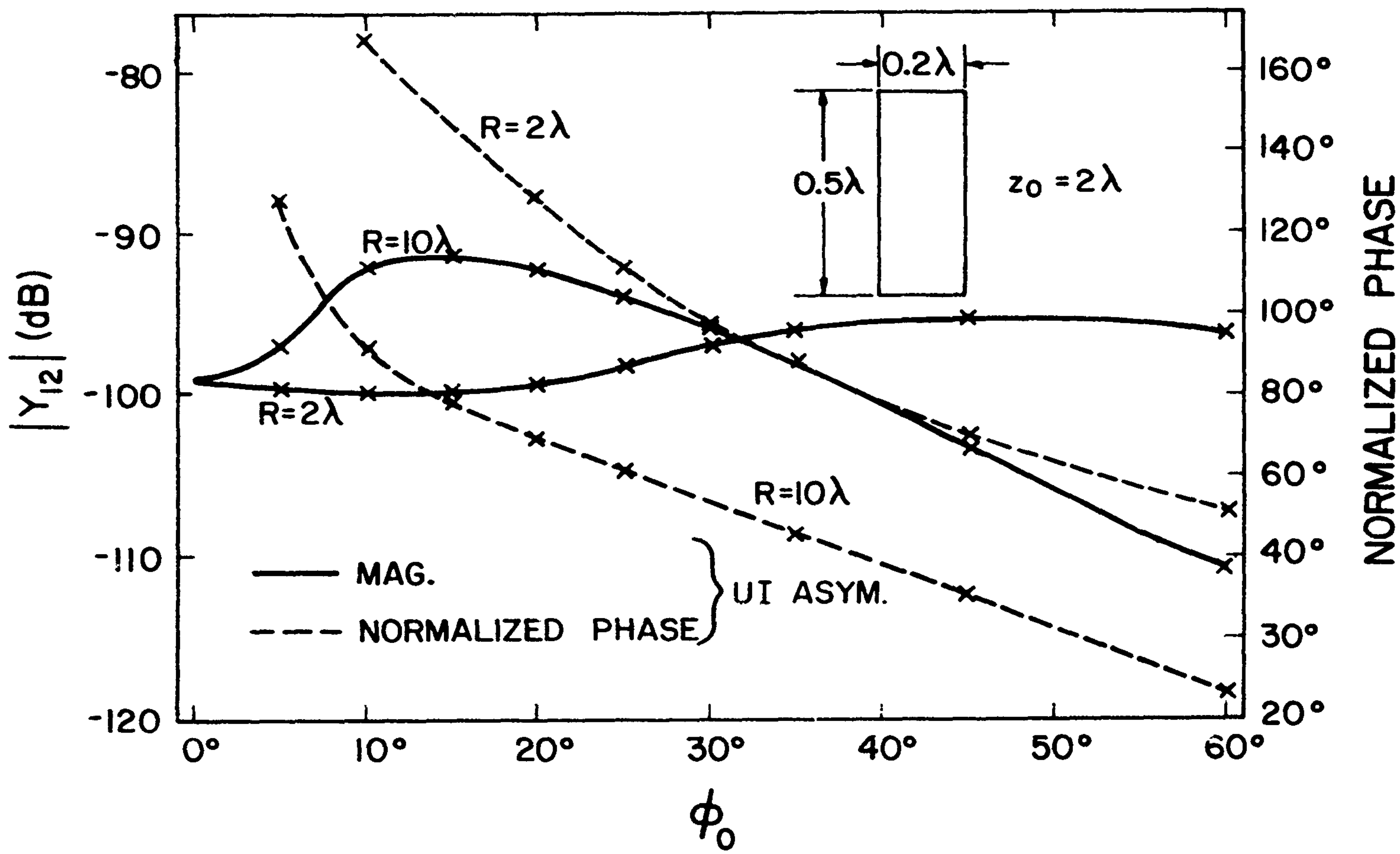

Figure F-2: Mutual admittance Y_{12} between two axial slots as a function of ϕ_0.

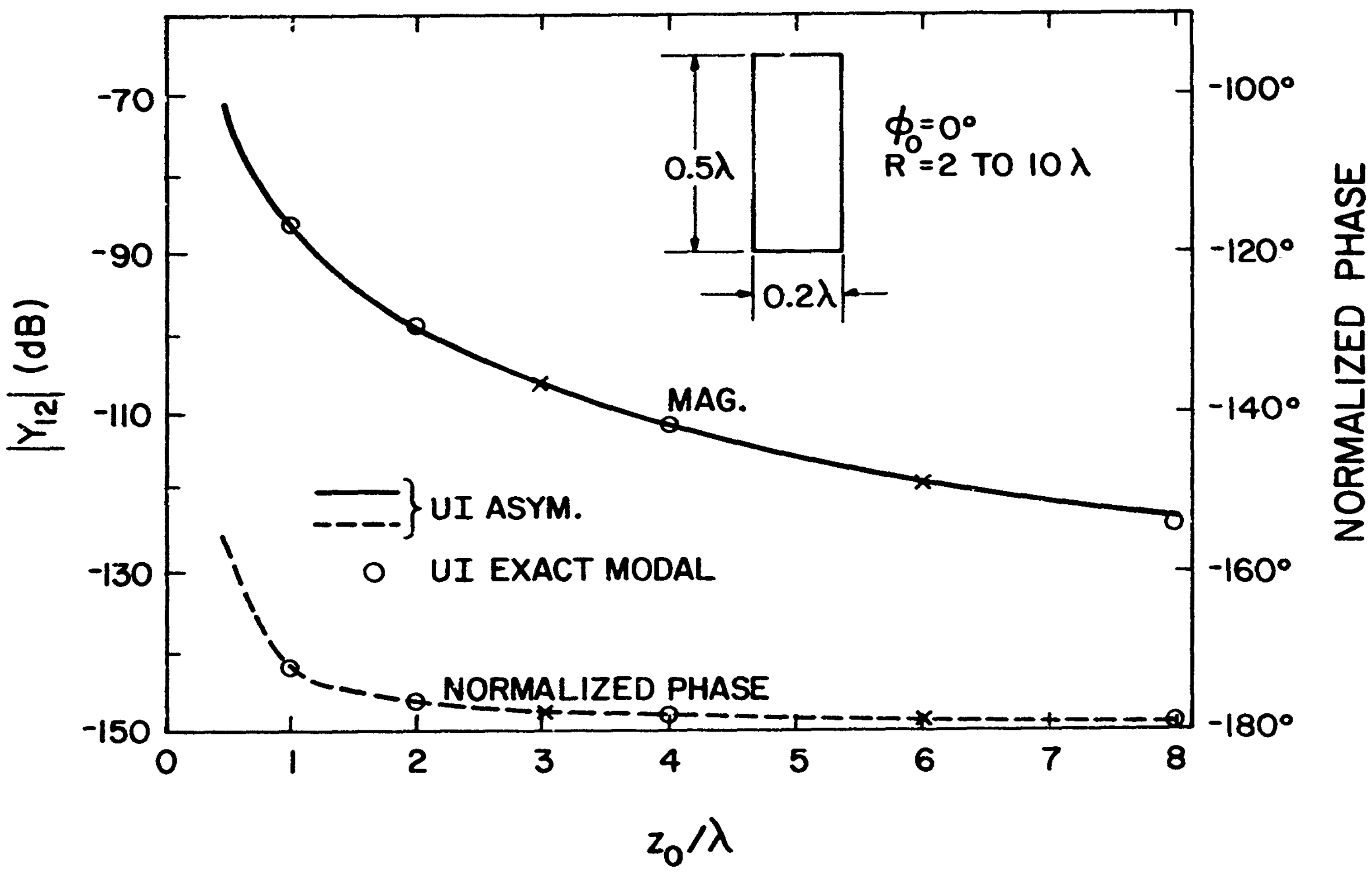

Figure F-3: Mutual admittance Y_{12} between two axial slots as a function of z_0.

78

APPENDIX B: COMPUTER PROGRAM LISTING

This appendix contains the program listing of all solutions, except the exact Hughes modal solution, discussed in the text.

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```
      PROGRAM CYORPL(INPUT,TAPE6,OUTPUT,TAPE7=OUTPUT,TAPE5=INPUT)
C
C
C
C     ***********************************************************
C     *                                                         *
C     *     MUTUAL ADMITTANCE OF SLOTS ON A CYLINDER OR PLANE   *
C     *     =================================================   *
C     *                                                         *
C     *                                                         *
C     *                                                         *
C     *                                                         *
C     *               BY:    S. W.  LEE                         *
C     *                      S. SAFAVI-NAINI                    *
C     *                      P. CHANG                           *
C     *                      C. L.  LAW                         *
C     *                                                         *
C     *                                                         *
C     *                                                         *
C     *             DATE:    8/10/77                            *
C     *                                                         *
C     *                                                         *
C     *             UNIVERSITY OF ILLINOIS                      *
C     *                                                         *
C     ***********************************************************
C
C
C THIS IS A MODIFIED VERSION OF THE PROGRAMS SOLVING MUTUAL
C ADMITTANCE OF SLOTS ON A CYLINDER OR ON A PLANE.
C THS PROGRAM IS MODIFIED TO RUN UNDER CDC CYBER 170 SERIES
C COMPUTER SYSTEMS.
C THE ORIGINAL VERSION WAS REPORTED IN "ELECTROMAGNETICS
C LABORATORY TECHNICAL REPORT NO.77-8"  DATE: MARCH 1977.
C
C THIS PROGRAM CONTAINS THE FOLLOWING SUBPROGRAMS:
C
C       (1) PLANAR
C            1A) EXACT               (PLANAR)
C            1B) APPROXIMATE         (APROX1 & APROX2)
C
C       (2)CYLINDRICAL
C            2A) UI ASYMPTOTIC       (CYLIND)
C            2B) UI EXACT MODAL      (PROG1 & PROG2)
C            2C) OSU ASYMPTOTIC      (CYLIND)
C            2D) PINY ASYMPTOTIC     (CYLIND)
C            2E) APPROXIMATE         (APROX1 & APROX2)
C            2F) UI(2) ASYMPTOTIC    (CYLIND)
C
C EACH SUBPROGRAM  IS A SUBROUTINE TO THE MAIN PROGRAM.
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
```

80

```
C THE INPUT FORMAT IS  AS FOLLOW :
C                                                                     **FORMAT**
C
C  --(1)/(2)                                                          *  A10
C  --1A/1B/2A/2B/2C/2D/2E/2F                                          * 2A10
C  --AXIAL/CIRCUMFERENTIAL                                            *  A10
C  --FREQUENCY                                                        *  G15.0
C  --SLOT DIMENSION (A,B)                                             * 2G15.0
C
C IF USING APPROXIMATE OR UI EXACT MODAL JUMP TO
C THE NEXT SECTION
C  --INTEGRATION GRID (IPA,IPB)                                       * 2I5
C
C IF USING OTHER THAN UI EXACT MODAL JUMP TO THE NEXT SECTION
C (UI EXACT MODAL IS USING TRAPEZOIDAL RULE FOR INTEGRATION.
C NCYCLE=NO. OF SUBSECTIONS BETWEEN ANY TWO SUCCESSIVE ZEROS
C        OF INTEGRAND
C MMAX IS THE MAXIMUM NO. OF TERMS WHICH HAS BEEN USED IN
C CALCULATION OF INFINITE SERIES)
C  --NCYCLE,MMAX                                                      * 2I5
C
C THE NORMALIZATION FACTOR (Y11)
C  --NORMALIZATION FACTOR                                             *  G15.0
C
C INPUT  ZO
C  --TOTAL NO. OF ELEMENTS IN  ZO   (MMAX.=20)                        *  I2
C  --ZO(1)                                                            *  G15.0
C  --ZO(2)                                                            *  G15.0
C  --  .                                                              *  G15.0
C
C IF USING PLANAR '(1)'  INPUT THE FOLLOWING ,IF NOT
C  JUMP TO THE NEXT  SECTION
C  --TOTAL NO. FO ELEMENTS IN   YO   (MAX.=20)                        *  I2
C  --YO(1)                                                            *  G15.0
C  --YO(2)                                                            *  G15.0
C  --  .                                                              *  G15.0
C
C IF USING CYLINDER '(2)'  INPUT THE FOLLOWING;
C  --TOTAL NO. OF ELEMENTS IN PHI   (MAX.=20)                         *  I2
C  --PHI(1)                                                           *  G15.0
C  --PHI(2)                                                           *  G15.0
C  --  .                                                              *  G15.0
C  --TOTAL NO. OF ELEMENTS IN RADIUS  (MAX.=20)                       *  I2
C  --RADIUS(1)                                                        *  G15.0
C  --RADIUS(2)                                                        *  G15.0
C  --   .                                                             *  G15.0
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT COMPLEX (C,H,Z),REAL(A-B,D-G,P-Y)
      REAL PHI(20),RADIUS(20),Z(20),YP(20)
      REAL TN(10),TNPI(10)
      INTEGER PINY,OSU,UI,TEST
```

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```
      LOGICAL CUM,AXIAL
      REAL MAG,Z0
      REAL CONF,CONF1,CONF2,KO,ZO
      COMMON PI,TZ1,TZ2,TY1,TY2,R,THETHA
      COMMON/DATA1/TN,TNPI,RHO,C1,C2,F2,IOP,CC,RADN,DEG
      COMMON/DATA4/ACO2,SN2,TN2,R2,ACON1,ACON2,CO1,C10
      COMMON/DATA2/AO,BO,ZO,YO
      COMMON/DATA3/KO,NCYCLE,PHIO,ZO,Y11,MMAX,A,B
      DATA TN/2.33811,4.08795,5.52156,6.78671,7.99417,
     $           9.02265,10.04017,11.00852,11.93602,12.82878/
      DATA TNPI/1.01879,3.24820,4.82010,6.16331,7.37218,
     $           8.48849,9.53545,10.52766,11.47506,12.38479/
      DATA UI/0/,OSU/0/,PINY/0/,TYPE1/10HEXACT     /,
     $     PRO2/10HCYLINDRICA/,PRO1/10HPLANAR    /,
     $     TYPE2/10HAPPROXIMAT/,TYPE3/10HUI EXACT M/,
     $     TYPE4/10HUI ASYMPTO/,TYPE5/10HOSU ASYMPT/,
     $     TYPE6/10HPINY ASYMP/,TYPE7/10HUI(2) ASYM/,
     $     CONF1/10HAXIAL     /,CONF2/10HCIRCUMFERE/
      DATA IPAA/1/,IPBB/1/
      PI=4*ATAN(1.E0)
      RADN=PI/180.   $   DEG=180./PI
      ACON1=187./64.   $   ACON2=2./3.
      CO1=(0.,1.)   $   C10=(1.,0.)
      WRITE(6,555)
      READ(5,9099)PRO
 9099 FORMAT(A10)
      IF(PRO.EQ.PRO1)GOTO 8082
      IF(PRO.EQ.PRO2)GOTO 8083
      WRITE(7,8084)
      STOP
 8083 IPLAN=2
      WRITE(6,666)
      GOTO8093
 8082 IPLAN=1
      WRITE(6,777)
 8093 READ(5,8081)TYPE,TYPEE
      READ(5,9099)CONF
      READ(5,8086)XK
      READ(5,8087)AA,BB
 8087 FORMAT(2G15.0)
      IF(TYPE.EQ.TYPE1)GOTO  8097
      IF(TYPE.EQ.TYPE2)GOTO 8099
      IF(TYPE.EQ.TYPE3)GOTO 9000
      IF(TYPE.EQ.TYPE4)GOTO 8097
      IF(TYPE.EQ.TYPE5)GOTO 8097
      IF(TYPE.EQ.TYPE6)GOTO 8097
      IF(TYPE.EQ.TYPE7)GOTO 8097
      WRITE(7,8084)
      STOP
 9000 READ(5,8098)NCYCLE,MMAX
      GOTO 8099
 8097 READ(5,8098)IPAA,IPBB
```

```
8099  READ(5,8086)Y11
      ZY1=CMPLX(Y11,0.E0)
      READ(5,8091)NDZ
      DO 8092 I=1,NDZ
8092  READ(5,8086)Z(I)
      IF(IPLAN.LT.2)GOTO 2230
      READ(5,8091)NDPHI
      DO 8095 I=1,NDPHI
8095  READ(5,8086)PHI(I)
      READ(5,8091)NDR
      DO 8094 I=1,NDR
8094  READ(5,8086)RADIUS(I)
      GOTO 8096
2230  READ(5,8091)NDY
      DO 8089 I=1,NDY
8089  READ(5,8086)YP(I)
8096  IF(CONF1.EQ.CONF)GOTO 9092
      AXIAL=.FALSE.
      CUM=.TRUE.
      GOTO 9093
9092  AXIAL=.TRUE.
      CUM=.FALSE.
9093  WRITE(6,555)
      WRITE(6,771)TYPE,TYPEE
771   FORMAT(///1X,20("*")/1X,"*"/1X,"*",3X,"METHOD OF SOLUTION :"
     $       ,4X,2A10/1X,"*"/1X,20("*"))
      WRITE(6,772)
772   FORMAT(/1X,20("*")/1X,"*")
773   FORMAT(1X,"*"/1X,20("*"))
      WRITE(6,888)XK
888   FORMAT(1X,"*",3X,"FREQUENCY :  K= ",E14.6)
      WRITE(6,773)
      WRITE(6,772)
      IF(IPLAN.EQ.1)GOTO 113
      WRITE(6,999)PRO2
999   FORMAT(1X,"*",3X,"GOMETRY :   ",A10)
      GOTO 112
113   WRITE(6,999)PRO1
112   WRITE(6,773)
      WRITE(6,772)
      WRITE(6,111)AA,BB
111   FORMAT(1X,"*",3X,"SLOT DIMENSION :  A(ALONG PHI)= ",F8.5,
     $       "  ;  B(ALONG Z)= ",F8.5)
      WRITE(6,773)
      WRITE(6,772)
      IF(CONF.EQ.CONF1)GOTO 114
      WRITE(6,333)
333   FORMAT(1X,"*",3X,"SLOT ORIENTATION :  CIRCUMFERENTIAL")
      GOTO 115
114   WRITE(6,444)
444   FORMAT(1X,"*",3X,"SLOT ORIENTATION :  AXIAL")
115   WRITE(6,773)
```

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```
      IF(TYPE.EQ.TYPE2)GOTO 774
      WRITE(6,772)
      IF(TYPE.EQ.TYPE3)GOTO 775
      WRITE(6,99)IPAA,IPBB
99    FORMAT(1X,"*",3X,"INTEGRATION GRID : ",I3," X",I3)
      GOTO 776
775   WRITE(6,881)NCYCLE,MMAX
881   FORMAT(1X,"*",3X,"NCYCLE= ",I5,"         ;      MMAX= ",I5)
776   WRITE(6,773)
774   WRITE(6,772)
      WRITE(6,882)Y11
882   FORMAT(1X,"*",3X,"NORMALIZATION FACTOR :  Y11= ",F7.2)
      WRITE(6,773)
      WRITE(6,883)
883   FORMAT(///10X,"$$$$$  DATA OUTPUT  $$$$$"//)
884   FORMAT(/2X,"PHI= ",F7.2," <DEG>    ;    ZO= ",F7.3,
     $      " ;    RADIUS= ",E14.5)
885   FORMAT(/2X,"    Y= ",F9.4,"      ;      Z= ",F7.3)
      IF(TYPE.EQ.TYPE3)GOTO 9095
      IF(TYPE.EQ.TYPE4)IOP=1
      IF(TYPE.EQ.TYPE5)IOP=2
      IF(TYPE.EQ.TYPE6)IOP=3
      IF(TYPE.EQ.TYPE7)IOP=4
      IF(CUM)GOTO 9094
      A=BB
      B=AA
      IPA=IPBB
      IPB=IPAA
      GOTO 9091
9094  A=AA
      B=BB
      IPA=IPAA
      IPB=IPBB
      GOTO 9091
9095  A=AA   $   B=BB    $   KO=XK
      DO 9096 I=1,NOR
      RHO=RADIUS(I)
      DO 9096 II=1,NDZ
      ZO=Z(II)
      DO 9096 III=1,NDPHI
      PHIO=PHI(III)*RADN
      WRITE(6,884)PHI(III),ZO,RHO
      IF(CONF1.EQ.CONF)GOTO  9097
      CALL PROG2(RHO,MAG,PHASE,DB,PHN)
      GOTO 9096
9097  CALL PROG1(RHO,MAG,PHASE,DB,PHN)
9096  WRITE(6,78)MAG,PHASE,DB,PHN
      STOP
8081  FORMAT(A10,A10)
8084  FORMAT(5X,"$$$  ERROR  $$$   ,PLEASE CHECK YOUR INPUT AGAIN")
8086  FORMAT(G15.0)
8091  FORMAT(I2)
```

```
8098  FORMAT(2I5)
C
9091  ICUM=1
      IAXIAL =2
      IF(.NOT.CUM) ICUM=2
      IF(.NOT.AXIAL) IAXIAL=1
      DO 30  IJ=ICUM,IAXIAL
      IF(IJ.EQ.2)GOTO 5
      A0=A*XK
      B0=B*XK
      GO TO 6
 5    SAVE=A
      A=B
      B=SAVE
      A0=A*XK
      B0=B*XK
      ITEMP=IPB
      IPB=IPA
      IPA=ITEMP
 6    WIDTH1=A0/IPA
      WIDTH2=B0/IPB
      C1=CEXP(CMPLX(0.E0,-PI/3.))
      C2=CEXP(CMPLX(0.E0,PI/4.))
      CC=C2**3
      F2=SQRT(PI)
      Y1=-A0/2.-WIDTH1/2.
      Z1=-B0/2.-WIDTH2/2.
      IF(IPLAN.EQ.1)GOTO 7
      GOTO 8
 7    NDR=1
      RADIUS(1)=1.E20
 8    CONTINUE
      DO 50 IRAD=1,NDR
      RHO=XK*RADIUS(IRAD)
      DO 60 JZ=1,NDZ
      Z0=Z(JZ)*XK
      IF(IPLAN.EQ.1) NDPHI=NDY
      DO 70 IY=1,NDPHI
      IF(IPLAN.EQ.1) GO TO 13
      Y0=RHO*PHI(IY)*RADN
      IF(PHI(IY).EQ.0.0) Y0=0.001
      Y=Y0/XK
      GO TO 14
 13   Y=YP(IY)
      Y0=Y*XK
 14   CONTINUE
      IF(TYPE.EQ.TYPE2)GOTO 17
      Y2=Y0-A0/2.-WIDTH1/2.
      Z2=Z0-B0/2.-WIDTH2/2.
      ZSUM=0.
      DO 80 K=1,IPA
      TY1=Y1+WIDTH1*K
```

```
      DO 90 L=1,IPB
      TZ1=Z1+WIDTH2*L
      DO 100 M=1,IPA
      TY2=Y2+WIDTH1*M
      DO 110 N=1,IPB
      TZ2=Z2+WIDTH2*N
      R=SQRT((TY2-TY1)**2+(TZ2-TZ1)**2)
      THETHA=ATAN2((TZ2-TZ1),(TY2-TY1))
      ACO2=COS(THETHA)**2
      R2=R**2
      IF(IPLAN.EQ.1) GO TO 600
      SN2=SIN(THETHA)**2
      TN2=TAN(THETHA)**2
      CALL CYLIND(IJ,ZSUM)
      GO TO 110
600   CALL PLANAR(IJ,ZSUM)
  110 CONTINUE
  100 CONTINUE
   90 CONTINUE
   80 CONTINUE
      ZY2=ZSUM*(WIDTH1*WIDTH2)**2*(-2.)/(AO*BO)
      IF(IPLAN.EQ.1)GOTO 26
      WRITE(6,884)PHI(IY),Z(JZ),RADIUS(IRAD)
      GOTO 18
26    WRITE(6,885)YP(IY),Z(JZ)
      GOTO 18
17    IF(IPLAN.EQ.1)GOTO 15
      WRITE(6,884)PHI(IY),Z(JZ),RADIUS(IRAD)
      GOTO 19
15    WRITE(6,885)YP(IY),Z(JZ)
19    IF(CONF.EQ.CONF1)GOTO 16
      CALL APROX1(AO,BO,ZO,YO,PI,ZY2)
      GOTO 18
16    CALL APROX2(AO,BO,ZO,YO,PI,ZY2)
18    MAG=CABS(ZY2)
      PHASE=ATAN2(AIMAG(ZY2),REAL(ZY2))*DEG
      ZEXPON=CEXP(CMPLX(0.E0,SQRT(YO**2+ZO**2)))
      ZPROD=ZY2*ZEXPON
      PHN=ATAN2(AIMAG(ZPROD),REAL(ZPROD))*DEG
      DB=20.*ALOG10(CABS(ZY2/ZY1))
      WRITE(6,78) MAG,PHASE,DB,PHN
   70 CONTINUE
   60 CONTINUE
      IF(IPLAN.EQ.1) GO TO 30
   50 CONTINUE
   30 CONTINUE
   78 FORMAT(1X,'Y12= ',E13.4,' <MHO>',F7.2, ' <DEG>;',5X,'DB= ',E12.5,
     1     ';      NORM PHASE= ',F7.2, ' <DEG>')
  555 FORMAT(/10X,'**********************************************'/)
  666 FORMAT(10X,'    MUTUAL ADMITTANCE OF SLOTS ON A CYLINDER    ')
  777 FORMAT(10X,'    MUTUAL ADMITTANCE OF SLOTS ON A PLANE    ')
      STOP
```

END

```
      SUBROUTINE APROX1(A0,B0,Z0,Y0,PI,ZY2)
      IMPLICIT COMPLEX (C,H,Z),REAL(A-B,D-G,P-Y)
      REAL TN(10),TNPI(10)
      REAL KA,Z0
      COMMON /DATA1/TN,TNPI,RHO,C1,C2,F2
      COMMON /CF/CVF,CUF,CV1F,CVPF,CUPF
      THETHA=ATAN2(Z0,Y0)
      IF(THETHA.GE.89.99*PI/180.) THETHA=89.99*PI/180.
      R=SQRT(Z0**2+Y0**2)
      RHOG=RHO/COS(THETHA)**2
      ZGR=(0.,-1.)*CEXP(CMPLX(0.E0,-R))/(240.*PI**2*R)
      KA=R*ABS((1./(2.*RHOG**2))**(1./3.))
      IF(KA.LT.0.7) GO TO 20
      CALL FOCK(KA)
      GO TO 30
20    CALL FOCK1(KA)
30    ZT1=CVF*(SIN(THETHA)**2+(0.,1.)*COS(2*THETHA)/R)
      ZT2=(0.,1.)*CUF*COS(THETHA)**2/R
      ZT3=(0.,1.)*(1./(SQRT(2.)*RHO*COS(THETHA)))**(2./3.)
     &    *SIN(THETHA)**4*CUPF
      HPHI=ZGR*(ZT1+ZT2+ZT3)
      IF(THETHA.EQ.0.) GO TO 40
      TM1=SIN(B0*SIN(THETHA)/2.)/(B0*SIN(THETHA)/2.)
      GO TO 50
40    TM1=1.
50    TM2=COS(A0*COS(THETHA)/2.)/(1.-(A0*COS(THETHA)/PI)**2)
      ZY2=-8.*A0*B0*(TM1*TM2/PI)**2*HPHI
      RETURN
      END
      SUBROUTINE APROX2(A0,B0,Z0,Y0,PI,ZY2)
      IMPLICIT COMPLEX (C,H,Z),REAL(A-B,D-G,P-Y)
      REAL TN(10),TNPI(10)
      REAL KA,Z0
      COMMON /DATA1/TN,TNPI,RHO,C1,C2,F2
      COMMON /CF/CVF,CUF,CV1F,CVPF,CUPF
      THETHA=ATAN2(Z0,Y0)
      IF(THETHA.GE.89.99*PI/180.) THETHA=89.99*PI/180.
      R=SQRT(Z0**2+Y0**2)
      RHOG=RHO/COS(THETHA)**2
      ZGR=(0.,-1.)*CEXP(CMPLX(0.E0,-R))/(240.*PI**2*R)
      KA=R*ABS((1./(2.*RHOG**2))**(1./3.))
      IF(KA.LT.0.7) GO TO 20
      CALL FOCK(KA)
      GO TO 30
20    CALL FOCK1(KA)
30    ZT1=CVF*(COS(THETHA)**2-(0.,1.)*COS(2.*THETHA)/R)
      ZT2=(0.,1.)*CUF*SIN(THETHA)**2/R
      HZ=ZGR*(ZT1+ZT2)
      TM1=(SIN(A0*COS(THETHA)/2.)/(A0*COS(THETHA)/2.))**2
      TM2=(COS(B0*SIN(THETHA)/2)/(1.-(B0*SIN(THETHA)/PI)**2))**2
      ZY2=-8.*A0*B0*TM1*TM2*HZ/PI**2
      RETURN
```

```
      END
      SUBROUTINE FMFN(X,N,FM,FN)
      REAL DUM1(400),DUM2(400)
      REAL FJ(400),XB,BSSY(400),FM(400),FN(400)
      PI=3.14159265
      XB=X
      IF(X-0.1) 10,10,20
   10 GAMLOG=ALOG(X/2.)+0.5772156649
      X2=X*X
      X3=X2*X
      X4=X*X3
      X5=X*X4
      BSSY1=2.*(GAMLOG*(1.-X2/4.+X4/64.)+X2/4.-3.*X4/128.)/PI
      BSSY2=-2./(PI*X)+2.*(GAMLOG*(X/2.-X3/16.+X5/384.)-X/4.+1.25*X3/16.
     &-3.33333*X5/768.)/PI
      GO TO 25
   20 CALL BESY(X,0,BSSY1,IER)
      CALL BESY(X,1,BSSY2,IER)
   25 CONTINUE
      BSSY(1)=BSSY1
      BSSY(2)=BSSY2
      DBSSY1=-BSSY(2)
      I=1
   80 I=I+1
      BSSY(I+1)=2.*(I-1)*BSSY(I)/X-BSSY(I-1)
      BSSYI1=BSSY(I+1)
      IF(ABS(BSSYI1).GE.1.E10) GO TO 100
      GO TO 80
  100 NMAX=I+1
      IF(NMAX.GE.N) NMAX=N
      N1=N-1
      CALL BSLJZ(XB,FJ,NMAX+1,0.D00,7,IERR,DUM1,DUM2)
      DFJ1=-FJ(2)
      FM(1)=1./(BSSY(1)**2+FJ(1)**2)
      FN(1)=1./(DBSSY1**2+DFJ1**2)
      DO 200 I=1,N1
      IF(I.GE.NMAX) GO TO 250
      DBSSY=BSSY(I)-I*BSSY(I+1)/X
      DFJ=FJ(I)-I*FJ(I+1)/XB
      FM(I+1)=1./(BSSY(I+1)**2+FJ(I+1)**2)
      FN(I+1)=1./(DBSSY**2+DFJ**2)
  200 CONTINUE
  250 CONTINUE
      N=NMAX
      RETURN
      END
C     SUBROUTINE 'BESY'
C        PURPOSE
C           COMPUTE THE Y BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESY(X,N,BY,IER)
```

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```
C
C        DESCRIPTION OF PARAMETERS
C           X   -THE ARGUMENT OF THE Y BESSEL FUNCTION DESIRED
C           N   -THE ORDER OF THE Y BESSEL FUNCTION DESIRED
C           BY  -THE RESULTANT Y BESSEL FUNCTION
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS NEGATIVE OR ZERO
C              IER=3  BY HAS EXCEEDED MAGNITUDE OF 10**70
C
C        REMARKS
C           VERY SMALL VALUES OF X MAY CAUSE THE RANGE OF THE LIBRARY
C           FUNCTION ALOG TO BE EXCEEDED
C           X MUST BE GREATER THAN ZERO
C           N MUST BE GREATER THAN OR EQUAL TO ZERO
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE
C           AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED
C           FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE
C           UNIVERSITY PRESS, 1958, P. 62
C
C     ..................................................................
C
      SUBROUTINE BESY(X,N,BY,IER)
C
C     CHECK FOR ERRORS IN N AND X
C
      IF(N)180,10,10
   10 IER=0
      IF(X)190,190,20
C
C     BRANCH IF X LESS THAN OR EQUAL 4
C
   20 IF(X-4.0)40,40,30
C
C        COMPUTE Y0 AND Y1 FOR X GREATER THAN 4
C
   30 T1=4.0/X
      T2=T1*T1
      P0=((((-.0000037043*T2+.0000173565)*T2-.0000487613)*T2
     1   +.00017343)*T2-.001753062)*T2+.3989423
      Q0=((((.0000032312*T2-.0000142078)*T2+.0000342468)*T2
     1   -.0000869791)*T2+.0004564324)*T2-.01246694
      P1=((((.0000042414*T2-.0000200920)*T2+.0000580759)*T2
     1   -.000223203)*T2+.002921826)*T2+.3989423
```

```
      Q1=((((-.0000036594*T2+.00001622)*T2-.0000398708)*T2
     1 +.0001064741)*T2-.0006390400)*T2+.03740084
      A=2.0/SQRT(X)
      B=A*T1
      C=X-.7853982
      Y0=A*P0*SIN(C)+B*Q0*COS(C)
      Y1=-A*P1*COS(C)+B*Q1*SIN(C)
      GO TO 90
C
C        COMPUTE Y0 AND Y1 FOR X LESS THAN OR EQUAL TO 4
C
   40 XX=X/2.
      X2=XX*XX
      T=ALOG(XX)+.5772157
      SUM=0.
      TERM=T
      Y0=T
      DO 70 L=1,15
      IF(L-1)50,60,50
   50 SUM=SUM+1./FLOAT(L-1)
   60 FL=L
      TS=T-SUM
      TERM=(TERM*(-X2)/FL**2)*(1.-1./(FL*TS))
   70 Y0=Y0+TERM
      TERM = XX*(T-.5)
      SUM=0.
      Y1=TERM
      DO 80 L=2,16
      SUM=SUM+1./FLOAT(L-1)
      FL=L
      FL1=FL-1.
      TS=T-SUM
      TERM=(TERM*(-X2)/(FL1*FL))*((TS-.5/FL)/(TS+.5/FL1))
   80 Y1=Y1+TERM
      PI2=.6366198
      Y0=PI2*Y0
      Y1=-PI2/X+PI2*Y1
C
C     CHECK IF ONLY Y0 OR Y1 IS DESIRED
C
   90 IF(N-1)100,100,130
C
C     RETURN EITHER Y0 OR Y1 AS REQUIRED
C
  100 IF(N)110,120,110
  110 BY=Y1
      GO TO 170
  120 BY=Y0
      GO TO 170
C
C   PERFORM RECURRENCE OPERATIONS TO FIND YN(X)
C
```

```
  130 YA=Y0
      YB=Y1
      K=1
  140 T=FLOAT(2*K)/X
      YC=T*YB-YA
      IF(ABS(YC)-1.0E70)145,145,141
  141 IER=3
      RETURN
  145 K=K+1
      IF(K-N)150,160,150
  150 YA=YB
      YB=YC
      GO TO 140
  160 BY=YC
  170 RETURN
  180 IER=1
      RETURN
  190 IER=2
      RETURN
      END
C     SUBROUTINE BSLJZ(X , FJ , NMAX , A , ND , IERR , FJAPRX , RR)
C
C THIS IS ONE OF THREE ROUTINES, "BSLJZ", "BSLIZ", AND "BSCJZ",
C BASED ON ALGORITHM 236 FROM "COMMUNICATIONS OF THE A.C.M.",
C AUGUST 1964. THIS ONE EVALUATES THE BESSEL FUNCTIONS OF THE
C FIRST KIND FOR REAL ORDERS AND NON-NEGATIVE REAL ARGUMENTS.
C
C THE PARAMETERS ARE DESCRIBED AS FOLLOWS, WITH "(I)", "(O)", AND
C "(I/O)" INDICATING, RESPECTIVELY, THAT A PARAMETER IS TO BE SET ON
C ENTRY, WILL BE SET BY THE ROUTINE, OR BOTH :
C
C *** ALL PARAMETERS EXCEPT "ND" , "IERR" , "NMAX" ARE ***
C *** SINGLE PRECISION REAL NUMBERS OR ARRAYS.         ***
C
C (I)   X        --  THE (NON-NEGATIVE) ARGUMENT TO THE BESSEL FUNCTIONS.
C (O)   FJ       --  AN ARRAY IN WHICH THE VALUES OF THE BESSEL FUNCTIONS
C                    ARE STORED, AS FOLLOWS: LET J(X;B) DENOTE THE VALUE
C                    OF THE BESSEL FUNCTION OF ORDER B WITH ARGUMENT X.
C                    THEN, FOR I = 1 TO ABS(NMAX)+1,
C                        FJ(I) = J(X;A + (I-1)*SIGN(NMAX)) .
C (I)   NMAX     --  REFER TO "FJ".
C (I)   A        --  REFER TO "FJ". NORMALLY, 0 <= A < 1, BUT THE ALGOR-
C                    ITHM WORKS, WITH SOME LOSS OF ACCURACY, FOR A >= 1.
C                    SEE THE PROGRAM NOTES BELOW.
C (I)   ND       --  THIS GIVES THE NUMBER OF SIGNIFICANT FIGURES OF
C                    ACCURACY DESIRED IN THE FUNCTION VALUES.
C (O)   IERR     -   THIS IS AN ERROR FLAG WHICH IS SET TO 0 IF THE
C                    INPUT PARAMETERS ARE OKAY, AND TO SOME POSITIVE
C                    VALUE IF ONE OF THE PARAMETERS IS INVALID. REFER
C                    TO THE ERROR EXITS AT THE END OF THE CODE FOR A
C                    DETAILED LIST OF THE VALUES OF IERR.
C (O)   FJAPRX   -   A SCRATCH ARRAY USED BY THE ROUTINE. IT MUST HAVE
```

```
C                       AT LEAST ABS(NMAX)+1 ENTRIES.
C (O)    RR        --   ANOTHER SCRATCH ARRAY. IT TOO MUST HAVE AT LEAST
C                       ABS(NMAX)+1 ENTRIES
C
C OTHER ROUTINES CALLED: ( * INDICATES A LOCAL ROUTINE )
C   *    NBS01Z -- INVERSE FUNCTION OF X*LOG(X)
C   *    UNDERZ -- ROUTINE TO CONTROL UNDERFLOW INTERRUPTS ON THE IBM 360.
C        MGAMMA -- GAMMA FUNCTION FROM THE IMSL LIBRARY.
C        ALOG   -- LOGARITHM
C        ABS    -- ABSOLUTE VALUE
C        MOD    -- REMAINDER
C        AMAX1  -- MAXIMUM OF 2 REALS
C
C NOTES:
C        THE METHOD OF COMPUTATION IS A VARIANT OF THE BACKWARD
C        RECURRENCE ALGORITHM OF J.C.P.MILLER (REFERENCE 1). THE
C        PURPORTED ACCURACY IS OBTAINED BY A JUDICIOUS SELECTION
C        OF THE INITIAL VALUE "NU" OF THE RECURSION INDEX (REP-
C        RESENTED IN THE CODE BY THE VARIABLE "XNU"), TOGETHER
C        WITH AT LEAST ONE REPETITION OF THE RECURSION WITH "NU"
C        REPLACED BY "NU"+5. NEAR A ZERO OF ONE OF THE BESSEL
C        FUNCTIONS, THE ACCURACY OF THAT PARTICULAR BESSEL FUNCTION
C        MAY DETERIORATE TO LESS THAN "ND" SIGNIFICANT DIGITS. THE
C        ALGORITHM IS MOST EFFICIENT WHEN X IS SMALL OR MODERATELY
C        LARGE.
C
C        THE ABOVE PARAGRAPH IS TAKEN FROM GAUTSCHI'S PRESENTATION
C        OF ALGORITHM 236 IN C.A.C.M. THE SELECTION OF THE INITIAL
C        "NU" IS DONE WITH THE AID OF THE FUNCTION NBS01Z, ALSO
C        BY GAUTSCHI (AND CALLED "T" BY HIM). IN THIS CODE, THE
C        FOLLOWING SPECIAL CASES HAVE BEEN ADDED:
C                A. X=0 WHEN NMAX > 0 OR A=0
C                B. A=0 AND NMAX < 0
C                C. A >= 1 : THE ALGORITHM WORKS IN THIS CASE, BUT THE
C                            INITIAL CHOICE OF "NU" IS NO LONGER
C                            OPTIMAL, AND SOME ACCURACY IS LOST. SIMPLE
C                            TESTS INDICATE THAT ONLY A FEW DECIMAL
C                            PLACES ARE SACRIFICED AT WORST. A LIMIT OF
C                            "ABIG" IS PLACED ON A TO AVOID OVERFLOW IN
C                            THE GAMMA FUNCTION. TO AVOID COMPLICATIONS,
C                            NMAX IS REQUIRED TO BE NON-NEGATIVE IF A > 1.
C
C REFERENCES:
C        1. GAUTSCHI, W. "RECURSIVE COMPUTATION OF SPECIAL FUNCTIONS",
C                UNIVERSITY OF MICHIGAN ENGINEERING SUMMER CONFER-
C                ENCES, NUMERICAL ANALYSIS, 1963.
C
C**********************************************************************
      SUBROUTINE BSLJZ(X , FJ , NMAX , A , ND , IERR , FJAPRX , RR)
      REAL NBS01Z
      DIMENSION FJ(1) , FJAPRX(1) , RR(1)
      LOGICAL   NEVEN , AFLAG
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```
      DATA      ONE/1D0/          , TWO/2D0/        , HALF/.5D0/      ,
     *          TEN/10D0/         , SMALL/1D-15/    , C1/.73576D0/    ,
     *          C2/1.3591D0/      , C3/2.3026D0/    , C4/1.3863D0/    ,
     *          ZERO/0D0/         , ABIG/55D0/      , TWOP5/2.5D0/    ,
     *          ALEPH/3777 0000 0000 0000 0000B/, FOUR/4D0/           ,
     *          C5/2000 4000 0000 0000 0000B/
C******
C INITIALIZE THE ERROR PARAMETER , TURN UNDERFLOW OFF , AND CHECK
C THE PARAMETERS FOR VALIDITY AND FOR THESE SPECIAL CASES:
C        A. X=0 WITH NMAX > 0 OR A=0
C        B. A=0 AND NMAX < 0
C
C THE CODE DELIBERATELY AVOIDS TESTING MORE THEN ONE THING IN EACH
C LOGICAL "IF" BELOW BECAUSE OF I.B.M. FORTRAN INEFFICIENCY IN THIS
C REGARD.
C
C IF A>1, NMAX MUST NOT BE NEGATIVE.
C******
      IERR = 0
C     CALL UNDERZ('OFF',SAVE)
      IF(A     .LT. ZERO) GOTO 999
      IF(A     .GT. ABIG) GOTO 998
      IF(X     .LT. ZERO) GOTO 997
      IF(NMAX  .GE. 0   ) GOTO 10
      IF(A     .EQ. ZERO) GOTO 10
      IF(A     .LE. SMALL) GOTO 996
      IF(A     .GE. ONE  ) GOTO 994
10    IF(X     .GT. ZERO) GOTO 40
      IF(NMAX  .GE. 0   ) GOTO 20
      IF(A     .GT. ZERO) GOTO 995
C******
C IF NMAX < 0, NMAXT IS SET HERE SO THAT ONLY J(X;A) IS CALCULATED.
C THE LOOP FOLLOWING STATEMENT 800 THEN CALCULATES THE REMAINING
C FUNCTIONS BY A SIMPLE RECURRENCE.
C IF A=0, NMAXT IS SET SO THAT J(X;A+N), N=0,...,-NMAX ARE
C CALCULATED; THE CODE AFTER 800 THEN REVERSES THE SIGN OF EVERY
C OTHER ONE.
C
C WE FIRST HANDLE THE CASE X=0.
C******
20    NTEMP = IABS(NMAX) + 1
      DO 30 I = 1,NTEMP
30    FJ(I) = ZERO
      IF(A .EQ. ZERO) FJ(1) = ONE
      GOTO 1000
C******
40    AFLAG = (A .EQ. ZERO) .AND. (NMAX .LT. 0)
      NMAXT = NMAX
      IF(NMAX .LT. 0) NMAXT = 1
      NTEMP = MAX0(NMAX+1,1)
      IF(.NOT. AFLAG) GOTO 60
      NMAXT = - NMAX
```

```
      NTEMP = NMAXT + 1
60    EPSLON = TEN**(-ND)/2
      DO 80 I = 1,NTEMP
80    FJAPRX(I) = ZERO
      CALL MGAMMA(ONE+A , RESULT , IER)
      SUM = (X/TWO)**A/RESULT
      D1  = C3*ND + C4
      R   = ZERO
      IF(NMAXT .GT. 0) R = NMAXT * NBS01Z(HALF*D1/NMAXT)
      S     = C2 * X * NBS01Z(C1*D1/X)
C******
C THE RECURSION INDEX "NU" IS DELIBERATELY CALCULATED AS A FLOATING
C POINT NUMBER RATHER THAN AN INTEGER, AND ALL COMPARISONS WITH IT
C ARE DONE AS FLOATING POINT COMPARISONS.
C******
      XNU   = ONE + AMAX1(R,S)
      XLIMIT = XNU/2
      TWOA = A + A
      XN    = ZERO
      FL    = ONE
C******
C THE OUTER ITERATION LOOP STARTS HERE.
C
C
C THE FOLLOWING LOOP IS DONE ENTIRELY IN FLOATING POINT FOR
C EFFICIENCY.
C******
200   XN = XN + ONE
      FL = FL * (XN + A)/(XN + ONE)
      IF(XN .LT. XLIMIT) GOTO 200
      OLDFL = FL
      OLDXN = XN
C
      N   = 2*XN
      XN = N
      NEVEN = .TRUE.
      R = ZERO
      S = ZERO
      TEMP1 = TWO/X
C******
C IN THE FOLLOWING LOOP, THE SUCCESSIVE VALUES OF "R" ARE PARTIAL
C FRACTIONS OF A CONTINUED FRACTION.
C******
300   DENOM = TEMP1 * (A + XN) - R
      IF(ABS(DENOM) .LE. SMALL) DENOM = DENOM + SMALL
      R = ONE/DENOM
      FLMBDA = ZERO
      IF(.NOT. NEVEN) GOTO 400
      FL      = FL * (XN + TWO)/(XN + TWOA)
      FLMBDA = FL * (XN + A)
400   S = R * (FLMBDA + S)
      IF(N .LE. NMAXT) RR(N) = R
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```
      N   = N - 1
      XN  = XN - ONE
      NEVEN = .NOT. NEVEN
      IF(N .GE. 1) GOTO 300
C******
      FJ(1) = SUM/(ONE + S)
      IF(NMAXT .EQ. 0) GOTO 600
      DO 500 N = 1,NMAXT
500   FJ(N+1) = RR(N) * FJ(N)
C******
C THE LATEST APPROXIMATIONS ARE CHECKED FOR IMPROVEMENT:
C******
600   DO 800 N = 1,NTEMP
        IF(ABS(FJ(N) - FJAPRX(N)) .LE. ABS(FJ(N))*EPSLON) GOTO 800
        DO 700 M = 1,NTEMP
700     FJAPRX(M) = FJ(M)
        XN     = OLDXN
        FL     = OLDFL
        XLIMIT = XLIMIT + TWOPS
        GOTO 200
800   CONTINUE
      IF(NMAX .GE. 0) GOTO 1000
C******
C IF NMAX<0, WE HAVE FINISHED OBTAINING J(X;A) , AND NOW
C ITERATE TO FIND ALL THE DESIRED FUNCTIONS.
C
C FIRST WE CHECK FOR THE SPECIAL CASE A=0.
C******
      IF(.NOT. AFLAG) GOTO 850
      NMAXT = -NMAX + 1
      DO 820 N = 2,NMAXT,2
820   FJ(N) = - FJ(N)
      GOTO 1000
C******
850   FJ(2) = TWO * A * FJ(1)/X - FJ(2)
      IF(NMAX .EQ. -1) GOTO 1000
C******
C THE FOLLOWING CODE IS A RENDITION OF THE LOOP
C                   DO 900 N = 2,NMAXT
C        900        FJ(N+1) = (2/X)*(A-N)*FJ(N) - FJ(N-1)
C
C WITH OVERFLOW DETECTION. AS SOON AS THE NUMBERS GET TOO BIG, THEY
C ARE SCALED DOWN (BY A POWER OF THE MACHINE BASE, SO AS TO AVOID
C LOSS OF PRECISION) AND THE CALCULATION CONTINUES. A SEPARATE LOOP
C TRANSFORMS THE SCALED VALUES TO THE CORRECT OUTPUT VALUES, SETTING
C TOO-LARGE ONES TO PLUS OR MINUS INFINITY.
C******
      NMAXT = -NMAX + 1
      FJN[illegible] = FJ([illegible])
      FJNM1 = FJ([illegible])
      TEMP1 = TWO/X
      OVER  = ZERO
```

```
      XNM1   = TWO
C
      DO 880 N = 3,NMAXT
      FJN    = TEMP1 * (A - XNM1) * FJNM1    -    FJNM2
      FJNM2 = FJNM1
      FJNM1 = FJN
      FJ(N) = FJN
      XNM1   = XNM1 + ONE
      RR(N) = OVER
      IF(ABS(FJN) .LT. C5) GOTO 880
      OVER   = OVER + ONE
      FJNM1 = FJNM1/C5
      FJNM2 = FJNM2/C5
880   CONTINUE
C
      IF(NMAXT .LE. 3) GOTO 1000
      OVER   = ZERO
      SCALE = ONE
C
      DO 900 N = 4,NMAXT
      IF(OVER .LT. FOUR) GOTO 890
      FJ(N) = SIGN(ALEPH,FJ(N))
      GOTO 900
890   IF(RR(N) .GT. OVER) SCALE = SCALE * C5
      FJ(N) = FJ(N) * SCALE
      OVER   = RR(N)
900   CONTINUE
      GOTO 1000
C******
C ERROR EXITS FOLLOW. MEANINGS OF THE EXIT VALUES OF "IERR" ARE:
C        0 : NO ERROR
C        1 : A < 0
C        2 : A > ABIG
C        3 : X < 0
C        4 : NMAX < 0 AND 0 < A < SMALL
C        5 : X=0, NMAX < 0, AND A > 0
C        6 : NMAX < 0 AND A >= 1
C******
994   IERR = IERR + 1
995   IERR = IERR + 1
996   IERR = IERR + 1
997   IERR = IERR + 1
998   IERR = IERR + 1
999   IERR = IERR + 1
1000  CONTINUE
C     CALL UNDERZ('S',SAVE)
      RETURN
      END
      REAL FUNCTION NBS01Z(X)
C******
C THIS IS A NUCLEUS FOR THE THREE BESSEL FUNCTION ROUTINES
C "BSLJZ" , "BSLIZ" , "BSCJZ" BASED ON ALGORITHM 236 FROM
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```
C "COMMUNICATIONS OF THE A.C.M.".
C IT EVALUATES THE INVERSE FUNCTION OF X*LOG(X) FOR X >= 1 TO AN
C ACCURACY OF ABOUT ONE PER CENT.
C FOR THE INTERVAL 0 <= X <= 10 A FIFTH DEGREE APPROXIMATION IS
C USED, OBTAINED BY TRUNCATING AN EXPANSION IN CHEBYCHEV POLYNOMIALS.
C FOR X > 10, A DIFFERENT APPROXIMATION IS GIVEN, AS CAN BE SEEN.
C******
      DATA      C1/.000057941D0/        , C2/-.00176148D0/      ,
     *          C3/.0208645D0/          , C4/-.129013D0/        ,
     *          C5/.85777D0/            , C6/1.10125D0/         ,
     *          ALPHA/.775D0/           , TEN/10D0/
      IF(X .GT. TEN) GO TO 10
      NBS01Z = ((((C1*X + C2)*X + C3)*X + C4)*X + C5)*X + C6
      RETURN
10    TEMP1 = ALOG(X)-ALPHA
      TEMP2 = (ALPHA-ALOG(TEMP1))/(1+TEMP1)
      NBS01Z = X/((1+TEMP2)*TEMP1)
      RETURN
      END
      SUBROUTINE PROD1(RHO,AMPY,PHASEY,AMPYDB,PHASNM)
C     PROGRAM TO COMPUTE THE MUTUAL ADMITTANCE BETWEEN TWO IDENTICAL
C     AXIAL SLOTS ON A CYLINDER ( UT MODAL SOLUTION)
      REAL    KO,KZ,KT,I2,KZKTRO
      COMPLEX  I1,Y12,PSIEXP,YN12
      REAL     F1(400), FM(400),FN(400),AIMAG,REAL,ATAN2
      COMMON/DATA3/KO,NCYCLE,PHIO,ZO,Y11,MMAX,A,B
C     INPUT PARAMETERS:
C     KO=WAVE NUMBER IN FREE SPACE IN TERMS OF 1/INCH
C     NCYCLE=NO. OF SUBSECTIONS BETWEEN ANY TWO  SUCCESSIVE ZEROS OF INTEGRAND
C     IN TRAPEZOIDAL RULE FOR NUMERICAL INTEGRATION
C     A*B= SLOT DIMENSION  B>A      <INCH>
C     RHO=RADIUS OF CYLINDER  <INCH>
C     PHIO=ANGULAR SEPARATION OF THE SLOTS (CENTER TO CENTER) <RADIAN>
C     ZO= SEPARATION OF THE SLOTS IN Z-DIRECTION   <INCH>
C     Y11= NORMALIZATION FACTOR
C     MMAX= MAXIMUM NO. OF TERMS WHICH HAS BEEN USED IN CALCULATION OF
C     INFINITE SERIES
      PI=3.14159265
      YO=1./(120.*PI)
      FREQ=3.E10*KO/(2.*PI*2.54)
      AKA=KO*A
      BKB=KO*B
      RK=KO*RHO
C     PHIA=HALF ANGULAR WIDTH OF THE SLOT
      PHIA=2.*ASIN(A/(2.*RHO))
C     COMPUTATION OF INFINITE SERIES
      MMAX12=MMAX+1
      DO 100 M=1,MMAX12
      M1=M-1
      IF(M.EQ.1) GO TO 99
      F1(M)=COS(M1*PHIO)*(SIN(M1*PHIA/2.)/(M1*PHIA/2.))**2
      GO TO 100
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   99 F1(M)=0.5
  100 CONTINUE
C     INTEGRATION OF PSI(KZ)*R1(M,KZ)*EXP(-J*KZ*ZO) BETWEEN 0 AND KO
C     DELTA= NEIGHBOURHOOD OF THE SINGULAR POINT KZ=KO IN WHICH THE INTEGRAL
C     HAS BEEN CALCULATED ANALYTICALLY
      DELTA=1.E-7*KO
C     NSECT1=NO. OF SAMPLES IN THE INTERVAL (0.,KO-DELTA).
      NSECT1=(IFIX((B+ZO+RHO)*KO/PI)+2)*NCYCLE
      DKZ=(KO-DELTA)/NSECT1
      NSECT=NSECT1+1
      I1=(0.,0.)
C     I1=FIRST INTEGRAL (BETWEEN 0. AND KO)
      DO 200 I=1,NSECT
      KZ=(I-1)*DKZ
      IF(KZ.EQ.0) KZ=0.00001*KO
      CIN=1.
      IF((I.EQ.1).OR.(I.EQ.NSECT)) CIN=0.5
      KT=SQRT(KO*KO-KZ*KZ)
      IF(ABS(KZ*B/2.-PI/2.).LE.1.E-8) KZ=1.000001*KZ
      PSIEXP=(COS(KZ*B/2.)/((KZ*B/2.)**2-(PI/2.)**2))**2*CIN*DKZ
     &*CEXP((0.,-1.)*KZ*ZO)
      MMAX1=MMAX
      ROKT=RHO*KT
C     COMPUTATION OF FM(N)=1./(JN(X)**2+YN(X)**2) AND FN(N)=1./(DJN(X)**2+
C     DYN(X)**2) FOR X=ROKT AND N=0 TO MMAX1 ; WHERE MMAX1 IS A NUMBER AFTER
C     WHICH  THE CONTRIBUTIONS OF FM(N) AND FN(N) TO THE INFINITE SUM
C     BECOME NEGLIGIBLE. MMAX1 IS A FUNCTION OF  THE ARGUMENT X AND IS ALWAYS
C     LESS THAN OR EQUAL TO MMAX. MMAX1, FM(N) AND FN(N) ARE CALCULATED
C     BY SUBROUTINE FMFN(X,MMAX1,FM,FN).
      CALL FMFN(ROKT,MMAX1,FM,FN)
      DO 200 M=1,MMAX1
      M1=M-1
  200 I1=I1+FN(M)*PSIEXP*F1(M)
C     COMPUTATION OF I2 (BETWEEN ZERO AND ETAMAX ; WHERE ETAMAX  IS A NUMBER
C     AFTER WHICH THE INTEGRAND BECOMES VERY SMALL)
      I2=0.
      ETAMAX=14./(ZO-B)
C     THE INTEGRATION IS CARRIED OUT BY TRAPEZOIDAL RULE. AT FIRST THE WHOLE
C     RANGE OF INTEGRATION (0.,ETAMAX) IS DEVIDED INTO TWO SUBINTERVALS :
C     (0.,ETA1) AND (ETA1,ETAMAX) , WHERE ETA1=ETAMAX/2.. THEN THE NUMERICAL
C     COMPUTATION OF THE  INTEGRAL IS PERFORMED IN THESE SUBINTERVALS WITH THE
C     NO. OF SAMPLES IN THE FIRST SUBINTERVAL TWO TIMES THAT IN THE SECOND ONE.
      ETA1=7./(ZO-B)
      NSECT1=(IFIX(SQRT(KO*KO+ETA1**2)*RHO/PI)+2)*NCYCLE
      DETA1=ETA1/NSECT1
      DETA2=2.*DETA1
      NSECT2=IFIX((ETAMAX-ETA1)/DETA2)+1
      NSECT=NSECT1+NSECT2+2
      DO 300 I=1,NSECT
      IF(I.LE.NSECT1+1) GO TO 220
      ETA=ETA1+(I-NSECT1-2)*DETA2
      DETA=DETA2
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```
      GO TO 240
  220 ETA=(I-1)*DETA1
      IF(ETA.EQ.0.) ETA=0.0001/A
      DETA=DETA1
  240 CIN=1.
      IF((I.EQ.1).OR.(I.EQ.NSECT1+1).OR.(I.EQ.NSECT1+2).OR.(I.EQ.NSECT)
     &) CIN=0.5
      PSEX=(COSH(ETA*B/2.)/((ETA*B/2.)**2+(PI/2.)**2))**2*DETA*CIN
     &*EXP(-ETA*ZO)
      KT=SQRT(KO*KO+ETA**2)
      MMAX1=MMAX
      CALL FMFN(RHO*KT,MMAX1,FM,FN)
      DO 300 M=1,MMAX1
      M1=M-1
  300 I2=I2+FN(M)*PSEX*F1(M)
      Y12=(I1+(0.,1.)*I2)*A*B*YO/(PI*KO*RHO**2)
C     NORMALIZATION OF THE PHASE OF Y12
      YN12=Y12*CEXP((0.,1.)*(KO*SQRT(ZO*ZO+(RHO*PHIO)**2)))
C     COMPUTATION OF THE ACTUAL PHASE 'PHASEY' AND NORMALIZED PHASE 'PHASNM'
C     OF Y12.
      PHASEY=ATAN2(AIMAG(Y12),REAL(Y12))*180./PI
      PHASNM=ATAN2(AIMAG(YN12),REAL(YN12))*180./PI
C     COMPUTATION OF THE MAGNITUDE OF THE Y12 IN TERMS OF <MHO> AND <DB>.
      AMPY=CABS(Y12)
      AMPYDB=ALOG10(AMPY/ABS(Y11))*20.
      RPHIK=KO*RHO*PHIO
      ZOK=KO*ZO
      PHIOD=PHIO*180./PI
      RETURN
      END
      SUBROUTINE PROG2(RHO,AMPY,PHASEY,AMPYDB,PHASNM)
C     PROGRAM FOR COMPUTATION OF THE MUTUAL ADMITTANCE BETWEEN TWO
C     IDENTICAL CIRCUMFERENTIAL SLOTS ON A CYLINDER(U1 MODAL SOLUTION)
      REAL    KO,KZ,KT,I2,KZKTRO
      COMPLEX  I1,Y12,PS1EXP,YN12
      REAL     F1(400), FM(400),FN(400),AIMAG,REAL,ATAN2
      COMMON/DATA3/KO,NCYCLE,PHIO,ZO,Y11,MMAX,A,B
C     INPUT PARAMETERS :
C     KO=WAVE NUMBER IN FREE SPACE IN TERMS OF 1/INCH
C     RHO=RADIUS OF CYLINDER <INCH>
C     PHIO=ANGULAR SEPARATION OF THE SLOTS (CENTER TO CENTER) <RADIAN>
C     ZO= SEPARATION OF THE SLOTS IN Z-DIRECTION   <INCH>
C     Y11= NORMALIZATION FACTOR
C     MMAX= MAXIMUM NO. OF TERMS WHICH HAS BEEN USED IN CALCULATION OF
C     INFINITE SERIES
C     NCYCLE=NO. OF SUBSECTIONS BETWEEN ANY TWO  SUCCESSIVE ZEROS OF INTEGRAND
C     IN TRAPEZOIDAL RULE FOR NUMERICAL INTEGRATION
      PI=3.14159265
      YO=1./(120.*PI)
      FREQ=3.E10*KO/(2.*PI*2.54)
      AKA=KO*A
      BKB=KO*B
```

```
      RK=KO*RHO
C     PHIB=HALF ANGULAR WIDTH OF THE SLOT
      PHIB=ASIN(A/(2.*RHO))
C     COMPUTATION OF INFINITE SERIES
      MMAX12=MMAX+1
      DO 100 M=1,MMAX12
      M1=M-1
      EPM=1.
      IF(M.EQ.1) EPM=2.
      PHIB1=PHIB
      IF(ABS(PHIB*M1-PI/2.).LE.1.E-7) PHIB1=PHIB*1.001
  100 FI(M)=COS(M1*PHIO)*(-PI*COS(M1*PHIB1)/(((M1*PHIB1)**2-(PI/2.)**2
     &)))**2*(1./EPM)
C     INTEGRATION OF PSI(KZ)*R1(M,KZ)*EXP(-J*KZ*ZO) BETWEEN 0 AND KO
C     DELTA= NEIGHBOURHOOD OF THE SINGULAR POINT KZ=KO IN WHICH THE INTEGRAL
C     HAS BEEN CALCULATED ANALYTICALLY
      DELTA=0.0001*KO
C     DELTA1= NEIGHBOURHOOD OF THE SINGULAR POINT KZ=KO WHERE THE INTEGRAND
C     VARIES RAPIDLY AND 'NDELTA' SAMPLES HAVE BEEN USED.
      DELTA1=0.01*KO
      NDELTA=100
      DKZ2=(DELTA1-DELTA)/NDELTA
C     NSECT1= NO. OF SUBSECTIONS BETWEEN  0 AND KO-DELTA1
      NSECT1=(IFIX((B+ZO+RHO)*KO/PI)+2)*NCYCLE
      DKZ1=(KO-DELTA1)/NSECT1
      NSECT=NSECT1+NDELTA+2
C     I1=FIRST INTEGRAL (BETWEEN 0. AND KO)
      I1=(0.,0.)
      DO 200 I=1,NSECT
      IF(I.LE.NSECT1+1) GO TO 120
      KZ=KO-DELTA1+(I-NSECT1-2)*DKZ2
      DKZ=DKZ2
      GO TO 140
  120 KZ=(I-1)*DKZ1
      IF(KZ.EQ.0) KZ=0.00001*KO
      DKZ=DKZ1
  140 CIN=1.
      IF((I.EQ.1).OR.(I.EQ.NSECT1+1).OR.(I.EQ.NSECT1+2).OR.(I.EQ.NSECT)
     &) CIN=0.5
      KT=SQRT(KO*KO-KZ*KZ)
      PSIEXP=(SIN(KZ*B/2.)/(KZ*B/2.))**2*CIN*DKZ*CEXP((0.,-1.)*KZ*ZO)
      MMAX1=MMAX
      ROKT=RHO*KT
C     COMPUTATION OF FM(N)=1./(JN(X)**2+YN(X)**2) AND FN(N)=1./(DJN(X)**2+
C     DYN(X)**2) FOR X=ROKT AND N=0 TO MMAX1 ; WHERE MMAX1 IS A NUMBER AFTER
C     WHICH  THE CONTRIBUTIONS OF FM(N) AND FN(N) TO THE INFINITE SUM
C     BECOME NEGLIGIBLE. MMAX1 IS A FUNCTION OF  THE ARGUMENT X AND IS ALWAYS
C     LESS THAN OR EQUAL TO MMAX. MMAX1, FM(N) AND FN(N) ARE CALCULATED
C     BY SUBROUTINE FMFN(X,MMAX1,FM,FN).
      CALL FMFN(ROKT,MMAX1,FM,FN)
      KZKTRO=(KZ/(KT*KO*RHO))**2
      DO 200 M=1,MMAX1
```

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      M1=M-1
      R1=(1./KT**2)*(FM(M)+M1**2*KZKTRO*FN(M))
  200 I1=I1+R1*PSIEXP*F1(M)
      I1=(2.*KO/(PI*RHO))*(I1-F1(1)*CEXP((0.,-1.)*KO*ZO)*(PI*PI/(2.*KO))
     &*(SIN(KO*B/2.)/(KO*B/2.))**2/(2.*(0.5772156649+ALOG(RHO*SQRT
     &(KO/2.)))+ALOG(DELTA)))
C     COMPUTATION OF I2 (BETWEEN ZERO AND ETAMAX ; WHERE ETAMAX  IS A NUMBER
C     AFTER WHICH THE INTEGRAND BECOMES VERY SMALL)
      I2=0.
      ETAMAX=14./(ZO-B)
C     THE INTEGRATION IS CARRIED OUT BY TRAPEZOIDAL RULE. AT FIRST THE WHOLE
C     RANGE OF INTEGRATION (0.,ETAMAX) IS DEVIDED INTO TWO SUBINTERVALS :
C     (0.,ETA1) AND (ETA1,ETAMAX) , WHERE ETA1=ETAMAX/2.. THEN THE NUMERICAL
C     COMPUTATION OF THE  INTEGRAL IS PERFORMED IN THESE SUBINTERVALS WITH THE
C     NO. OF SAMPLES IN THE FIRST SUBINTERVAL TWO TIMES THAT IN THE SECOND ONE.
      ETA1=7./(ZO-B)
      NSECT1=(IFIX(SQRT(KO*KO+ETA1**2)*RHO/PI)+2)*NCYCLE
      DETA1=ETA1/NSECT1
      DETA2=2.*DETA1
      NSECT2=IFIX((ETAMAX-ETA1)/DETA2)+1
      NSECT=NSECT1+NSECT2+2
      DO 300 I=1,NSECT
      IF (I.LE.NSECT1+1) GO TO 220
      ETA=ETA1+(I-NSECT1-2)*DETA2
      DETA=DETA2
      GO TO 240
  220 ETA=(I-1)*DETA1
      IF(ETA.EQ.0.) ETA=0.0001/B
      DETA=DETA1
  240 CIN=1.
      IF((I.EQ.1).OR.(I.EQ.NSECT1+1).OR.(I.EQ.NSECT1+2).OR.(I.EQ.NSECT)
     &) CIN=0.5
      PSEX=(SINH(ETA*B/2.)/(ETA*B/2.))**2*EXP(-ETA*ZO)*DETA*CIN
      KT=SQRT(KO*KO+ETA**2)
      ETKTRO=(ETA/(RHO*KO*KT))**2
      MMAX1=MMAX
      CALL FMFN(RHO*KT,MMAX1,FM,FN)
      DO 300 M=1,MMAX1
      M1=M-1
      R1=(1/(KT*KT))*(FM(M)-M1*M1*ETKTRO*FN(M))
  300 I2=I2+F1(M)*R1*PSEX
      I2=I2*2.*KO/(PI*RHO)
      Y12=(I1+(0.,1.)*I2)*A*B*Y0/(2.*PI*PI*RHO)
C     NORMALIZATION OF THE PHASE OF Y12
      YN12=Y12*CEXP((0.,1.)*(KO*SQRT(ZO*ZO+(RHO*PHIO)**2)))
C     COMPUTATION OF THE ACTUAL PHASE 'PHASEY' AND NORMALIZED PHASE 'PHASNM'
C     OF Y12.
      PHASEY=ATAN2(AIMAG(Y12),REAL(Y12))*180./PI
      PHASNM=ATAN2(AIMAG(YN12),REAL(YN12))*180./PI
C     COMPUTATION OF THE MAGNITUDE OF THE Y12 IN TERMS OF <MHO> AND <DB>.
      AMPY=CABS(Y12)
      AMPYDB=ALOG10(AMPY/ABS(Y11))*20.
```

102

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```
      RPHIK=KO*RHO*PHIO
      ZOK=KO*ZO
      PHIOD=PHIO*180./PI
      RETURN
      END
C
C  THIS SUBROUTINE IS USED TO CALCULATE THE FUNCTIONS CVF,CUF,CV1F,CVPF,CUPF
C
      SUBROUTINE FOCK(X)
      IMPLICIT REAL(A-B,D-H,P-Y),COMPLEX (C,Z)
      REAL TN(10),TNPI(10)
      COMMON/CF/CVF,CUF,CV1F,CVPF,CUPF
      COMMON/DATA1/TN,TNPI,RHO,C1,C2,F2,IOP,CC,RADN,DEG
      F1=SQRT(X)
      F3=X**1.5
      CVF=0.
      CUF=0.
      CV1F=0.
      CVPF=0.
      CUPF=0.
      DO 20 N=1,10
      ZTN=TN(N)*C1
      ZTNPI=TNPI(N)*C1
      C3=CEXP(CMPLX(0.0,-X)*ZTNPI)
      C4=CEXP(CMPLX(0.0,-X)*ZTN)
      CVF=CVF+C3/ZTNPI
      CUF=CUF+C4
      CV1F=CV1F+C3
      CVPF=(1.0-CMPLX(0.0,2*X)*ZTNPI)*C3/ZTNPI+CVPF
      CUPF=(1.0-CMPLX(0.0,2.*X/3.)*ZTN)*C4+CUPF
   20 CONTINUE
      CVF=F2*F1*CVF/C2
      CUF=2.*F2*F3*C2*CUF
      CV1F=2.*F2*F3*C2*CV1F
      CVPF=F2*CVPF/(2.*F1*C2)
      CUPF=3.*F2*F1*C2*CUPF
      RETURN
      ENTRY FOCK1
      XTHREE=X**3
      F1=SQRT(X)
      F3=XTHREE
      Z1=F2*C2*SQRT(F3)
      Z2=CMPLX(0.0,1.0/60.)*XTHREE
      Z3=F2*X**4.5/(C2*64.)
      F4=F3**2
      CVF=1.0-Z1/4.+7*Z2+7.*Z3/8.-4.141E-3*F4
      CUF=1.0-Z1/2.+25.*Z2+5.*Z3-3.701E-2*F4
      CV1F=1.0+Z1/2.-35.*Z2-7.*Z3+4.555E-2*F4
      CVPF=.375*F1*F2/CC+21.*Z2/X+63.*Z3/(16.*X)-2.485E-2*F4/X
      CUPF=.75*F1*F2/CC+CMPLX(0.0,1.25*X**2)+22.5*Z3/X
     1  -2.221E-1*F4/X
```

103

```
      RETURN
      END
```



```
C
      SUBROUTINE PLANAR(IJ,ZSUM)
      IMPLICIT COMPLEX(C,H,Z),REAL(A-B,D-G,P-Y)
      REAL ZO
      COMMON PI,TZ1,TZ2,TY1,TY2,R,THETHA
      COMMON/DATA2/A,B,ZO,YO
      GO TO (10,20),IJ
   10 XM1=((TZ2-TZ1)/R)**2
      XM2=2.-3.*XM1
      XRL=XM2/R
      XIM=XM1+XM2/R**2
      HA=CEXP(CMPLX(0.E0,-R))*CMPLX(XRL,-XIM)/(240.*R*PI**2)
      FACTOR=COS(PI*TY1/A)*COS(PI*(TY2-YO)/A)
      ZSUM=ZSUM+FACTOR*HA
      RETURN
   20 ZA1=CMPLX(1./R**2,1./R)*(2.-3.*ACO2)
      ZA2=CEXP(CMPLX(0.E0,-R))*(ACO2+ZA1)/R
      HA=(0.,-1.)*ZA2/(240.*PI**2)
      FACTOR=COS(PI*TZ1/B)*COS(PI*(TZ2-ZO)/B)
      ZSUM=ZSUM+FACTOR*HA
      RETURN
      END
C
C THIS SUBROUTINE IS USED TO GET THE "CYLINDRICAL" SOLUTION
C
      SUBROUTINE CYLIND(IJ,ZSUM)
      IMPLICIT COMPLEX(C,H,Z),REAL(A-B,D-G,P-Y)
      REAL ZO,KA
      REAL TN(10),TNPI(10)
      COMMON PI,TZ1,TZ2,TY1,TY2,R,THETHA
      COMMON/CF/CVF,CUF,CV1F,CVPF,CUPF
      COMMON/DATA1/TN,TNPI,RHO,C1,C2,F2,IOP,CC,RADN,DEG
      COMMON/DATA2/A,B,ZO,YO
      COMMON/DATA4/ACO2,SN2,TN2,R2,ACON1,ACON2,C01,C10
      ZOR=(0.,-1.)*CEXP(CMPLX(0.E0,-R))/(240.*R*PI**2)
      IT=0
      ANGLE=ATAN2(ABS(TZ2-TZ1),ABS(TY2-TY1))*DEG
      IF(ANGLE.LT.89.99) GO TO 10
      IT=1
      THETHA=PI*89.99/180.
      ZW=C01/R+CEXP(CMPLX(0.E0,-PI/4.E0))*SQRT(PI*R/2.)/RHO
   10 RHOG=RHO/ACO2
      KA=R*ABS((1./(2.*RHOG**2))**(1./3.))
      IF(KA.LT.0.7) GO TO 20
      CALL FOCK(KA)
      GO TO 30
   20 CALL FOCK1(KA)
```

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```
   30  IF(IT.EQ.1) GO TO 40
       ZW=C01*(1./ACO2)/R*(CUF-CV1F*SN2)
   40  GO TO (21,22,23,24),IOP
   21  ZHB1=CMPLX(1.E0,-1./R)*CVF
       ZHB2=CUF/R**2
       ZHB3=C01/(SQRT(2.E0)*RHO)**ACON2
       ZHB4=ACO2**ACON2*CVPF+SN2*(CUPF/ACO2**(1./3.))
       HB=ZGR*(ZHB1-ZHB2+ZHB3*ZHB4)
       ZTM1=C01*CUPF/(SQRT(2.E0)*RHOG)**ACON2
       HT=C01*ZGR/R*(CVF+CMPLX(1.E0,-2./R)*CUF+ZTM1)
       IF(IJ.LT.2)GOTO 25
       HZ=HB*ACO2+HT*SN2
       GOTO 500
 25    HPHI=HB*SN2+HT*ACO2
       GO TO 500
   22  HB=ZGR*CVF
       HT=C01*ZGR*CUF/R
       IF(IJ.LT.2)GOTO 26
       HZ=HB*ACO2+HT*SN2
       GOTO 500
 26    HPHI=HB*SN2+HT*ACO2
       GO TO 500
   23  ZTM2=C01*(1.-3.*SN2)/R
       IF(IJ.LT.2)GOTO 27
       HZ=ZGR*CVF*(ACO2+C01*(2.-3.*ACO2)/R)
       GOTO 500
 27    HPHI=ZGR*(CVF*(SN2+ZTM2)+ZW)
       GOTO 500
 24    IF(IJ.LT.2)GOTO 28
       HZA=CVF*(ACO2*C10+(2.-3.*ACO2)*CMPLX(1./R2,1./R))
       HZB=C01*(CVPF/(SQRT(2.)*RHOG)**ACON2)
       HZC=CMPLX(-11./12.*ACO2,(-11./6.-2./3.*TN2+ACON1*ACO2)/R)
       HZ=ZGR*(HZA+HZB*HZC)
       GOTO500
 28    HPA=CVF*CMPLX(SN2+(2.-3.*SN2)/R2,(2.-3.*SN2)/R)
       HPQ=C01*(CUF-CVF)/ACO2/R
       HPC=C01/(SQRT(2.)*RHOG*ACO2)**ACON2
       HP1D=(CVPF/ACO2**(1./3.))
       HP2D=CMPLX(4./3.*SN2-11./12.*SN2*ACO2,
      $           .75*ACO2-7./12.*SN2+ACON1*SN2*ACO2)
       HP3D=C01/12./R*CUPF/ACO2**(1./3.)
       HP4D=HPC*(HP1D*HP2D+HP3D)
       HPHI=ZGR*(HPA+HPQ+HP4D)
  500  ZGREEN=HPHI
       IF(IJ.EQ.2)ZGREEN=HZ
       FACTOR=COS(PI*TY1/A)*COS(PI*(TY2-YO)/A)
       IF(IJ.EQ.2)FACTOR=COS(PI*TZ1/B)*COS(PI*(TZ2-ZO)/B)
       ZSUM=ZSUM+FACTOR*ZGREEN
       RETURN
       END
```

ATTACHMENT B

Report 77-13

SIMPLE APPROXIMATE FORMULA FOR MUTUAL ADMITTANCE
BETWEEN SLOTS ON A CYLINDER

ELECTROMAGNETICS LABORATORY
TECHNICAL REPORT NO. 77-13

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SIMPLE APPROXIMATE FORMULA FOR MUTUAL ADMITTANCE BETWEEN SLOTS ON A CYLINDER

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Mutual Admittance of Slots on Cylinder
GTD
Conformal Array of Slots

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Based on a newly developed asymptotic Green's function for a magnetic dipole on a conducting surface [1], this paper presents a simple, closed-form formula for the mutual admittance between two slots on a cylinder or a plane. When compared with the exact solution obtained by numerical integrations, this formula gives accurate results when the slots are relatively small and their separation large.

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110

SIMPLE APPROXIMATE FORMULA FOR MUTUAL ADMITTANCE BETWEEN SLOTS ON A CYLINDER*

S. W. Lee and S. Safavi-Naini†

ABSTRACT

Based on a newly developed asymptotic Green's function for a magnetic dipole on a conducting surface [1], this paper presents a simple, closed-form formula for the mutual admittance between two slots on a cylinder or a plane. When compared with the exact solution obtained by numerical integrations, this formula gives accurate results when the slots are relatively small and their separation large.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

This paper contains two results for the mutual admittance Y_{12} between two slots on the surface of a large conducting cylinder (including the conducting plane as a special case). The first and the main result is that an approximate, closed-form solution of Y_{12} is derived. This solution may be considered as a simplified version of the asymptotic solution of Y_{12} reported in [1], as the two surface integrals over the apertures of the slots are no longer needed in the present approximate solution. Our second result concerns the derivation of an exact solution of Y_{12}, which is given in terms of an inverse Fourier transform and an infinite summation of cylindrical modes. This solution is based on the original expression for Y_{12} described by Stewart, Golden, and Pridmore-Brown [2], [3], and is more suitable for numerical calculation for some cases.

This work is undertaken for the following reasons. The determination of Y_{12} (or its dual problem for Z_{12} between two dipoles) is not only a classical problem in electromagnetics that has attracted wide attention [1] - [10], but also an integral part in the design of modern conformal arrays [11] - [15]. In the latter application, Y_{12} must be repeatedly calculated for a large number of times. Thus, a simple closed-form solution should greatly reduce the computation effort and, furthermore, provide a better physical insight for the design problem as the "cause" and "effect" can be readily identified in a closed-form solution.

The organization of this paper is as follows. In Section 2, we first define Y_{12}, and then give the final form of its approximate solution. Discussions and numerical results are presented in Section 3. In the last two sections (4 and 5), the derivations of both the approximate and the exact modal solutions of Y_{12} are given. Fock functions used in the text are described in the Appendix.

2. APPROXIMATE FORMULA FOR MUTUAL ADMITTANCE

Referring to Figure 1, consider two slots on the surface of an infinitely long conducting cylinder with radius R. The orientation of the slots may be either circumferential (Figure 1b where $a_n > b_n$, $n = 1,2$), or axial (Figure 1c where $a_n < b_n$). The problem is to determine the mutual admittance between these two slots when kR is large.

First let us define mutual admittance. Throughout this work we always assume that

(i) the slots are thin, and (2.1a)

(ii) their length is roughly a half-wavelength. (2.1b)

Then the aperture field in each slot can be adequately approximated by a simple cosine distribution, which is the so-called "one-mode" approximation. For example, if slot 1 is circumferential (Figure 1b), its aperture field under the "one-mode" approximation is given by

$$\vec{E} = V_1\vec{e}_1 \; , \; \vec{H} = I_1\vec{h}_1 \tag{2.2a}$$

where
$$\vec{e}_1 = \hat{z}\sqrt{\frac{2}{a_1b_1}}\cos\frac{\pi}{a_1}y \quad , \quad \vec{h}_1 = \hat{x}\times\vec{e}_1 \tag{2.2b}$$

$$y = R\phi \; . \tag{2.2c}$$

(V_1, I_1) are respectively the modal (voltage, current) of slot 1. The mutual admittance Y_{12} is defined by

$$Y_{12} = Y_{21} = \frac{I_{21}}{V_1} \tag{2.3}$$

where I_{21} is the induced current in slot 2 when slot 1 is excited by a voltage V_1 and slot 2 is short-circuited. An alternative expression for Y_{12} is

$$Y_{12} = \frac{1}{V_1 V_2} \iint_{A_2} \vec{E}_2 \times \vec{H}_1 \cdot d\vec{s}_2 \tag{2.4}$$

where

A_2 = aperture of slot 2

$\vec{H}_1$ = magnetic field when slot 1 is excited with voltage V_1, and slot 2 is covered by a perfect conductor

$\vec{E}_2$ = electric field when slot 2 is excited with voltage V_2, and slot 1 is covered by a perfect conductor.

Because $\vec{H}_1 = I_{21}\vec{h}_2$ and $\vec{E}_2 = V_2\vec{e}_2$, it is a simple matter to verify that (2.3) and (2.4) are equivalent [16].

There is an alternative definition of mutual admittance. Instead of (2.2), a modal voltage $\bar{V}_1$ (with a bar) may be defined through the expression for the aperture field of slot 1 as follows:

$$\vec{E} = \hat{z}\,\frac{1}{b}\,\bar{V}_1 \cos\frac{\pi}{a_1}\,y \quad , \tag{2.5a}$$

or equivalently

$$\bar{V}_1 = \int_0^b (\hat{z} \cdot \vec{E})_{y=0}\, dz \quad . \tag{2.5b}$$

Then a different mutual admittance $\bar{Y}_{12}$ is defined by (2.4) after replacing (V_1,V_2) by $(\bar{V}_1,\bar{V}_2)$. It can be easily shown that

$$\bar{Y}_{12} = \frac{1}{2}\left(\frac{a_1 a_2}{b_1 b_2}\right)^{1/2} Y_{12} \quad . \tag{2.6}$$

Two remarks are in order: (i) In the limiting case that b_1 and $b_2 \to 0$, Y_{12} goes to zero as $(b_1 b_2)^{1/2}$, whereas $\bar{Y}_{12}$ approaches a constant independent of b_1 and b_2. (ii) For the special case $a_1 = a_2 = \lambda/2$ and

$R \to \infty$, it is $\bar{Y}_{12}$, not Y_{12}, that is identical to the mutual impedance Z_{12} between two corresponding dipoles calculated by the classical Carter's method [5], [8], [9]. (iii) When the slots are excited by waveguides (transmission lines), one often uses Y_{12} ($\bar{Y}_{12}$). From here on, we will concentrate on Y_{12} instead of $\bar{Y}_{12}$.

For the two slots in Figure 1, the final form of an approximate solution of Y_{12} is as follows (for exp $+j\omega t$ time convention):

Circumferential slots

$$Y_{12} \doteq -\frac{8}{\pi^2}(a_1b_1a_2b_2)^{1/2}\, S(b_1 \sin\theta)\, S(b_2 \sin\theta)\, C(a_1 \cos\theta)\, C(a_2 \cos\theta)\, \bar{g}_\phi \tag{2.7a}$$

Axial slots

$$Y_{12} \doteq -\frac{8}{\pi^2}(a_1b_1a_2b_2)^{1/2}\, S(a_1 \cos\theta)\, S(a_2 \cos\theta)\, C(b_1 \sin\theta)\, C(b_2 \sin\theta)\, \bar{g}_z \,. \tag{2.7b}$$

The various factors in (2.7) are explained below. S and C are simple trigonometric functions

$$S(x) = \frac{\sin(kx/2)}{(kx/2)} \quad , \quad C(x) = \frac{\cos(kx/2)}{1-(kx/\pi)^2} \,. \tag{2.8}$$

The (simplified) Green's functions $\bar{g}_\phi$ and $\bar{g}_z$ are given by

$$\bar{g}_\phi = G(s)\Big[v(\xi)\Big(\sin^2\theta + \frac{j}{ks}\cos 2\theta\Big) + \frac{j}{ks}u(\xi)\cos^2\theta + ju'(\xi)(\sqrt{2}\,kR\cos\theta)^{-2/3}\sin^4\theta\Big] \tag{2.9a}$$

$$\bar{g}_z = G(s)\Big[v(\xi)\Big(\cos^2\theta - \frac{j}{ks}\cos 2\theta\Big) + \frac{j}{ks}u(\xi)\sin^2\theta\Big] \tag{2.9b}$$

where

$$G(s) = \frac{k^2Y_0}{2\pi j}\frac{e^{-jks}}{ks} \quad , \quad Y_0 = \frac{1}{120\pi} \tag{2.10}$$

$$\xi = (k\cos^4\theta/2R^2)^{1/3}\, s \tag{2.11}$$

$$s = \sqrt{z_0^2 + (R\phi_0)^2} \tag{2.12}$$

$$\theta = \tan^{-1} (z_0/R\phi_0) \ . \tag{2.13}$$

The Fock functions u and v are explained in the Appendix. In the limiting case $kR \to \infty$ (slots on a planar surface), (2.9) is further simplified to become

$$\left.\begin{aligned} \bar{g}_\phi &= G(s)\left[\sin^2\theta + \frac{j}{ks}(2 - 3\sin^2\theta)\right] , \\ \bar{g}_z &= G(s)\left[\cos^2\theta + \frac{j}{ks}(2 - 3\cos^2\theta)\right] , \end{aligned}\right\} \quad kR \to \infty \ . \tag{2.14}$$

The formula in (2.4) is an approximate solution, valid under the condition

$$kR >> 1 \quad \text{and} \quad ks >> 1 \ . \tag{2.15}$$

The numerical accuracy of the formula is discussed in Section 3, and its derivation in Section 4.

3. NUMERICAL RESULTS AND DISCUSSION

For the two slots in Figure 1, the final form of the approximate solution of Y_{12} is given in (2.7). Generally speaking, its accuracy is good only if

(i) the size of the slots is small in terms of wavelength, and/or

(ii) the separation of the slots is large in terms of wavelength.

In this section, we will give some numerical examples to illustrate the quantitative accuracy of (2.7).

(A) <u>Circumferential Slot</u> - (Figures 2 and 3). The size of each slot is $0.5\lambda \times 0.2\lambda$, and the cylinder radius is 1λ. Y_{12} is presented in (dB, normalized phase) format, where $\mathrm{dB} = 20 \log_{10} (|Y_{12}|$ in mho) and normalized phase is equal to $\mathrm{Arg}(Y_{12} \exp jks)$. Three solutions of Y_{12} are given: the UI exact modal solution calculated from (5.2), (5.3) and (5.9); the UI asymptotic solution reported in [1]; and the approximate solution in (2.7). We note that all the three solutions are in an excellent agreement.

(B) <u>Percentage Error vs. Slot Position</u> - (Figures 4 and 5). In these figures, the coordinates of each point determine the center-to-center distance, in ϕ and z directions between two slots. The pairs of numbers in the parentheses are the percentage error in magnitude and the absolute error in phase of Y_{12} as calculated by the approximate formula, respectively. For the circumferential slots (Figure 4), the accuracy is generally very good. For the axial slots (Figure 5), the approximate formula gives erratic results (as high as 27 percent error in magnitude) when the two slots are very closely displaced in the ϕ-direction. The reason for this inaccuracy is that the surface field due to a magnetic dipole varies very rapidly as a function of z when the observation point is close by.

(C) Accuracy vs. Cylinder Radius (Figure 6). The accuracy of the approximate formula is not sensitive to the radius of the cylinder.

(D) Planar Slots (Tables 1 and 2). The mutual admittance Y_{12} between two identical slots of dimension ($a = 0.69\lambda$, $b = 0.3\lambda$) on an infinite conducting plane is calculated as a function of z_0 and y_0 (the center-to-center distance between two slots in z and y directions, see Figure 1b). Y_{12} is given in (dB, phase in degrees). In both E-plane and H-plane couplings, the approximate formula is accurate when the separation is at least two wavelengths (2.6"). It should be also remarked that the present slots ($0.69\lambda \times 0.3\lambda$) are relatively large. The accuracy of the approximate formula is better when the slots are smaller.

4. DERIVATION OF APPROXIMATE FORMULA

We will now give the derivation of the formula in (2.7a)[that of (2.7b) is very similar]. Consider a circumferential infinitesimal dipole located at Q' on the surface of a cylinder (Figure 7) which is described by the magnetic current density

$$\vec{K} = \hat{\phi}\,\frac{1}{R}\,\delta(r - R)\;\delta(\phi)\;\delta(z) \quad . \tag{4.1}$$

At an observation point Q on the cylinder, the ϕ-component of the $\vec{H}$ field, denoted by g_ϕ, is determined in Eq. (2.16b) of [1], which reads in the present notation,

$$\begin{aligned} g_\phi(t,\alpha) \sim G(t) \Bigg\{ & v(\xi)\left[\sin^2 \alpha + \frac{j}{kt}\cos 2\alpha\right] \\ & + \left(\frac{j}{kt}\right) u(\xi)\left[\cos^2 \alpha \left(1 - \frac{2j}{kt}\right) + \left(\frac{j}{kt}\right) \sin^2 \alpha\right] \\ & + j(\sqrt{2}\;kR/\cos^2 \alpha)^{-2/3} \\ & \cdot \left[v'(\xi)\,\sin^2 \alpha + \left(\tan^4 \alpha + \frac{j}{kt}\right) u'(\xi)\,\cos^2 \alpha\right]\Bigg\} \end{aligned} \tag{4.2}$$

where (t,α) are the cylindrical coordinates of Q with respect to the origin at Q' on a developed cylinder, and

$$\xi = (k \cos^4 \theta/2R^2)^{1/3}\, t \quad . \tag{4.3}$$

The formula in (4.2) is mainly based on a classical work of Fock [17], and contains a modification that introduces a field dependence on the surface curvature in the binormal direction of the surface ray (see Section 6 of [1]). This formula is asymptotically valid for $kR \to \infty$, and may be used to calculate the field at any point on the cylindrical surface.

Making use of the Green's function in (4.2), we next calculate the surface field H_ϕ due to slot 1 on a cylinder (Figure 8). The aperture distribution of slot 1 is described in (2.2a), which may be replaced by an equivalent magnetic current density (p. 108 of [18])

$$\vec{K} = \hat{\phi}\ \delta(r - R) \sqrt{\frac{2}{a_1 b_1}}\ V_1 \cos\ (\pi y/a_1) \quad . \tag{4.4}$$

Then, H_ϕ at an observation point Q is obtained by superposition, namely,

$$H_\phi(Q) = \sqrt{\frac{2}{ab}}\ V_1 \iint_{A_1} \left(\cos \frac{\pi}{a_1} y\right) g_\phi(t,\alpha)\ dy\ dz \quad . \tag{4.5}$$

The expression for calculating the mutual admittance Y_{12} between the two slots in Figure 8 is given in (2.4). Note that $\vec{E}_2$ is described much as (2.2a) and $\vec{H}_1$ in (4.5). Then (2.4) becomes

$$Y_{12} = \frac{-2}{\sqrt{a_1 b_1 a_2 b_2}} \iint_{A_1} dy\ dz \iint_{A_2} dy_2\ dz_2 \left(\cos \frac{\pi}{a_1} y\right)\left(\cos \frac{\pi}{a_2} y_2\right) g_\phi(t,\alpha) \quad . \tag{4.6}$$

The distance t in (4.6) is given by

$$t = [(s \cos \theta + y_2 - y)^2 + (s \sin \theta + z_2 - z)]^{1/2} \quad . \tag{4.7}$$

If s is large relative to the length of either slot, t may be approximated by

$$t \simeq \begin{cases} s & \text{(4.8a)} \\ s\left(1 + \cos \theta\ \dfrac{y_2 - y}{s} + \sin \theta\ \dfrac{z_2 - z}{s}\right) \quad . & \text{(4.8b)} \end{cases}$$

In evaluating the magnitude of g_ϕ in (4.6), we use the approximation in (4.8a), whereas in evaluating its progressive phase term, we use (4.8b).

Then the integrals in (4.6) can be explicitly carried out. After a further approximation by dropping the terms of order $(ks)^{-3} = (kt)^{-3}$ in (4.2), we obtain the desired solution of Y_{12} in (2.7a).

5. EXACT MODAL SOLUTION

The admittance Y_{12} defined in (2.3) may be calculated exactly by using cylindrical modes, as has been done by Stewart, Golden and Pridmore-Brown [2], [3]. Extensive numerical results of Y_{12} calculated from the SGP solution are reported in [13], [14]. As will be explained below, the SGP solution is not suitable for numerical calculations when the slot separation z_0 (Figure 1a) is large. In this section, we will derive an alternative modal solution of Y_{12} which does not have this difficulty.

Let us first consider the circumferential slots shown in Figure 1b. For the case that $a_1 = a_2 = a$ and $b_1 = b_2 = b$ (identical slots), the mutual admittance Y_{12} is given in Eq. (8) of [3]*, which reads in the present notation,

$$Y_{12} = \int_{-\infty}^{\infty} dk_z \sum_{m=-\infty}^{\infty} \psi(m,k_z) G(m,k_z) e^{-j(m\phi_0 + k_z z_0)} \tag{5.1a}$$

where

$$\psi(m,k_z) = \frac{ab}{8\pi^2 R} \frac{\sin^2(k_z b/2)}{(k_z b/2)^2} \cdot \left\{ \frac{\sin\,(m\phi_a + \pi/2)}{(m\phi_a + \pi/2)} + \frac{\sin\,(m\phi_a - \pi/2)}{(m\phi_a - \pi/2)} \right\}^2 \tag{5.1b}$$

$$\phi_a = (a/2R)$$

$$G(m,k_z) = Y_0 \left[\frac{jk}{k_t} \frac{H_m^{(2)\prime}(k_t R)}{H_m^{(2)}(k_t R)} + \left(\frac{mk_z}{k_t^2 R} \right)^2 \frac{k_t}{jk} \frac{H_m^{(2)}(k_t R)}{H_m^{(2)\prime}(k_t R)} \right] . \tag{5.1c}$$

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* The multiplication factor 2 in the definition of ϕ_b in [3] is a misprint and should be removed.

$$k_t = \begin{cases} \sqrt{k^2 - k_z^2} & , \text{ if } k \geq k_z \\ -j\sqrt{k_z^2 - k^2} & , \text{ if } k \leq k_z \end{cases} .$$

Rewrite Y_{12} in terms of its real and imaginary parts:

$$Y_{12} = G + jB \quad . \tag{5.2}$$

It can be shown that G is given by

$$G = \int_0^k \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \cos k_z z_0 \; \psi(m,k_z) R(m,k_z) \; dk_z \tag{5.3a}$$

where

$$R(m,k_z) = \frac{2}{\pi k_t R} \cdot \frac{k}{k_t} \cdot \left[\frac{1}{M_m^2(k_t R)} + \left(\frac{mk_z}{k_t k \, R} \right)^2 \frac{1}{N_m^2(k_t R)} \right] \tag{5.3b}$$

$$M_m^2(\chi) = J_m^2(\chi) + Y_m^2(\chi) \tag{5.3c}$$

$$N_m^2(\chi) = J_m'^2(\chi) + Y_m'^2(\chi) \tag{5.3d}$$

$$\varepsilon_m = \begin{cases} 2 \; , & m = 0 \\ 1 \; , & m \neq 0 \end{cases} . \tag{5.3e}$$

We note that G contains a *finite* integral and can be evaluated in a straightforward manner by standard numerical integration techniques. The imaginary part of Y_{12} is given by

$$B = \int_{C_1} \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \cdot \cos k_z z_0 \cdot \psi(m,k_z) \cdot W(m,k_z) \; dk_z \tag{5.4a}$$

where the integration contour C_1 is shown in Figure 9 and

$$W(m,k_z) = \begin{cases} \dfrac{k}{k_t}\,(J_m J'_m + Y_m Y'_m)\left[\dfrac{1}{M_m^2(k_t R)} - \left(\dfrac{mk_z}{k_t kR}\right)^2 \dfrac{1}{N_m^2(k_t R)}\right], & \text{if } k > k_z \\[2ex] \dfrac{-k}{|k_t|}\left[\dfrac{K'_m(|k_t|R)}{K_m(|k_t|R)} - \left(\dfrac{mk_z}{|k_t|kR}\right)^2 \dfrac{K_m(|k_t|R)}{K'_m(|k_t|R)}\right], & \text{if } k < k_z \end{cases} \tag{5.4b}$$

The computation of B as given in (5.4a) can be quite laborious because (i) the integration with respect to k_z is of infinite range, and the factor cos $k_z z$ is highly oscillatory for large kz_0, (ii) $W(m,k_z)$ has nonintegrable singularities of opposite sign on both sides of $k_z = k$ (iii) $W(m,k_z)$ decays slowly with respect to m and k_z.

To circumvent the above difficulties in evaluating B, we adopt a method introduced by Duncan [19] in the study of cylindrical antenna problems. Let us rewrite (5.4a)

$$B = \mathrm{Im}\left\{\sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m}\left[-j\int_{C_1} F(m,k_z)\,\sin k_z z_0\, dk_z + \int_{C_1} F(m,k_z)e^{jk_z z_0}\, dk_z\right]\right\} \tag{5.5}$$

where

$$F(m,k_z) = [R(m,k_z) + jW(m,k_z)]\psi(m,k_z) \quad . \tag{5.6}$$

The imaginary part of the first term inside the bracket of (5.5) is

$$\mathrm{Im}\left\{-j\int_{C_1} F(m,k_z)\,\sin k_z z_0\, dk_z\right\} = -\int_0^k R(m,k_z)\psi(m,k_z)\,\sin k_z z_0\, dk_z \quad . \tag{5.7}$$

In order to compute the imaginary part of the second term of (5.5), the integration contour C_1 is deformed into C_2 (Figure 9) according to the theory of complex variables. This manipulation leads to

$$\mathrm{Im}\int_{C_1} F(m,k_z)e^{jk_z z_0}\, dk_z = \mathrm{Im}\int_{C_2} F(m,k_z)e^{jk_z z_0}\, dk_z \quad . \tag{5.8}$$

Make the change of variable $k_z = j\eta$ in (5.8). Substitution of the resultant equation and (5.7) into (5.5) gives

$$B = \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \left\{ -\int_0^k R(m,k_z)\psi(m,k_z) \sin k_z z_0 \, dk_z \right.$$
$$\left. + \int_0^{\infty} R(m,j\eta)\psi(m,j\eta) e^{-\eta z_0} \, d\eta \right\} . \tag{5.9}$$

Our final expression for Y_{12} is given in (5.2), with its real part G in (5.3) and its imaginary part B in (5.9). Several remarks are in order: (i) Not only G but also B is determined by $R(m,k_z)$, which is much simpler than $W(m,k_z)$ defined in (5.4b). (ii) B contains only a finite integral. (iii) The infinite integral in B, i.e., the second integral in (5.9a), contains an exponentially decaying factor $\exp[-z_0 - a)\eta]$ in its integrand. The emergence of the evaluation of B is faster for larger z_0. This is in contrast to the original expression of Y_{12} given in (5.1). (iv) There is no nonintegrable singularity in (5.3) or (5.9).

The same method applies to the derivation of an alternative expression of Y_{12} for two identical axial slots (Figure 1c with $a_1 = a_2 = a$ and $b_1 = b_2 = b$). We give below only the final result:

$$Y_{12} = -\frac{abY_0}{\pi k R^2} \sum_{m=0}^{\infty} \frac{\cos m\phi_0}{\varepsilon_m} \left[\int_0^k \Phi(m,k_z) e^{-jk_z z_0} \frac{dk_z}{N_m^2(k_t R)} \right.$$
$$\left. + j \int_0^{\infty} \Phi(m,j\eta) e^{-\eta z_0} \frac{d\eta}{N_m^2\left(R\sqrt{\eta^2 + k^2}\right)} \right] \tag{5.10a}$$

where

$$\Phi(m,k_z) = \left[\frac{\sin(m\phi_a)}{(m\phi_a)} \cdot \frac{\cos(k_z b/2)}{(k_z b/2)^2 - (\pi/2)^2} \right]^2 . \tag{5.10b}$$

In summary, the alternative expression of the exact modal solutions is given in (5.2), (5.3), and (5.9) for two identical circumferential slots, and in (5.10) for two identical axial slots.

APPENDIX

FOCK FUNCTIONS

In this appendix we define and list some useful formulas of the functions $w_1(t)$, $w_2(t)$, $v(\xi)$, $u(\xi)$, and $v_1(\xi)$. These functions are commonly known as Fock functions.

(i) Definition: For a complex t and a real ξ,

$$w_1(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} dz \exp\left(tz - \frac{1}{3} z^3\right) \tag{A-1}$$

$$w_2(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_2} dz \exp\left(tz - \frac{1}{3} z^3\right) = w_1^*(t) \tag{A-2}$$

$$v(\xi) = \frac{1}{2} e^{j\pi/4} \xi^{1/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} \frac{w_2(t)}{w_2'(t)} e^{-j\xi t}\, dt \tag{A-3}$$

$$u(\xi) = e^{j3\pi/4} \xi^{3/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} \frac{w_2'(t)}{w_2(t)} e^{-j\xi t}\, dt \tag{A-4}$$

$$v_1(\xi) = e^{j3\pi/4} \xi^{3/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} t \frac{w_2(t)}{w_2'(t)} e^{-j\xi t}\, dt \tag{A-5}$$

where integration contour $\Gamma_1(\Gamma_2)$ goes from ∞ to 0 along the line $\text{Arg } z = -2\pi/3$ $(+2\pi/3)$ and from 0 to ∞ along the real axis. Because of different time conventions, $w_1(w_2)$ above is equal to $w_2(w_1)$ defined in [17].

(ii) Residue series representation: For real positive ξ,

$$v(\xi) = e^{-j\pi/4}\sqrt{\pi}\ \xi^{1/2} \sum_{n=1}^{\infty} (t_n')^{-1} e^{-j\xi t_n'} \tag{A-6}$$

$$u(\xi) = e^{j\pi/4} 2\sqrt{\pi}\ \xi^{3/2} \sum_{n=1}^{\infty} e^{-j\xi t_n} \tag{A-7}$$

$$v_1(\xi) = e^{j\pi/4} 2\sqrt{\pi}\ \xi^{3/2} \sum_{n=1}^{\infty} e^{-j\xi t_n'} \tag{A-8}$$

$$v'(\xi) = \frac{1}{2} e^{-j\pi/4}\sqrt{\pi}\ \xi^{-1/2} \sum_{n=1}^{\infty} (1 - j2\xi t_n')(t_n')^{-1} e^{-j\xi t_n'} \tag{A-9}$$

$$u'(\xi) = e^{j\pi/4} 3\sqrt{\pi}\ \xi^{1/2} \sum_{n=1}^{\infty} \left(1 - j\,\frac{2}{3}\,\xi t_n\right) e^{-j\xi t_n} \tag{A-10}$$

where $\{t_n\}$ and $\{t_n'\}$ are zeros of $w_2(t)$ and $w_2'(t)$, respectively, and are tabulated in [17] and [1].

(iii) Small argument asymptotic expansion: For real positive ξ and $\xi \to 0$,

$$v(\xi) \sim 1 - \frac{\sqrt{\pi}}{4} e^{j\pi/4}\xi^{3/2} + \frac{7j}{60}\xi^3 + \frac{7\sqrt{\pi}}{512} e^{-j\pi/4}\xi^{9/2} - 4.141 \times 10^{-3}\xi^6 + \ldots \tag{A-11}$$

$$u(\xi) \sim 1 - \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} + \frac{5j}{12}\xi^3 + \frac{5\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} - 3.701 \times 10^{-2}\xi^6 + \ldots \tag{A-12}$$

$$v_1(\xi) \sim 1 + \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} - \frac{7j}{12}\xi^3 - \frac{7\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} + 4.555 \times 10^{-2}\xi^6 + \ldots \tag{A-13}$$

$$v'(\xi) \sim \frac{3\sqrt{\pi}}{8} e^{-j3\pi/4}\xi^{1/2} + \frac{7j}{20}\xi^2 + \frac{63\sqrt{\pi}}{1024} e^{-j\pi/4}\xi^{7/2} - 2.485 \times 10^{-2}\xi^5 + \ldots \tag{A-14}$$

$$u'(\xi) \sim \frac{3}{4}\sqrt{\pi}\ e^{-j3\pi/4}\xi^{1/2} + \frac{5j}{4}\xi^2 + \frac{45\sqrt{\pi}}{128} e^{-j\pi/4}\xi^{7/2} - 2.221 \times 10^{-1}\xi^5 + \ldots \tag{A-15}$$

(iv) Numerical evaluation: For $\xi \geq \xi_0$, the residue series representation with the first ten terms in the summation may be used. For $\xi \leq \xi_0$, the small argument asymptotic expansion with the first five terms may be used. It has been indicated in [12] that the smoothest crossover is obtained if $\xi_0 = 0.6$. In the present study, we set $\xi_0 = 0.7$, where the difference in the two representations is less than 0.1% in magnitude and 0.9° in phase [1].

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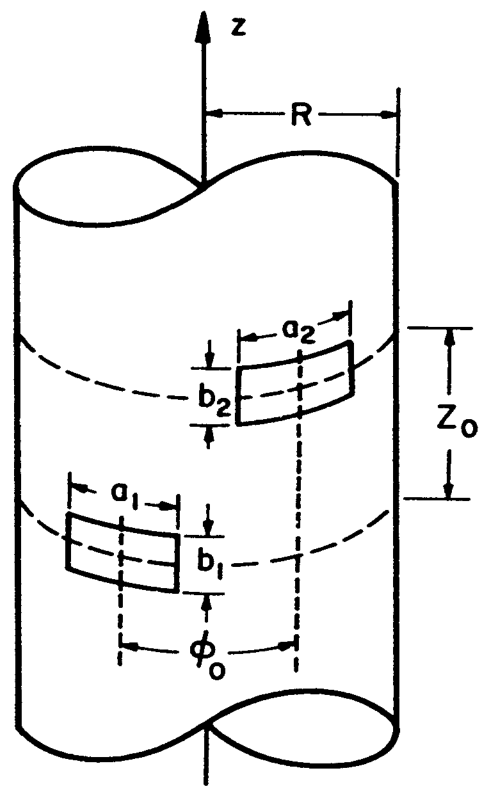
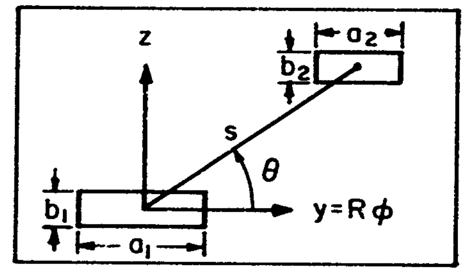
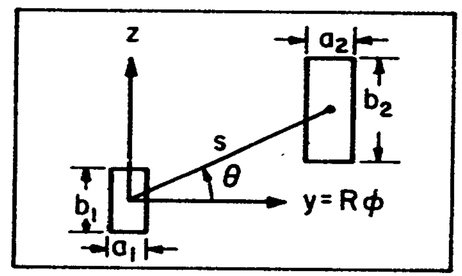


(a) Three-dimensional view



(b) Circumferential slots on developed cylinder.



(c) Axial slots on developed cylinder.

Figure 1. Two slots on the surface of a conducting cylinder.

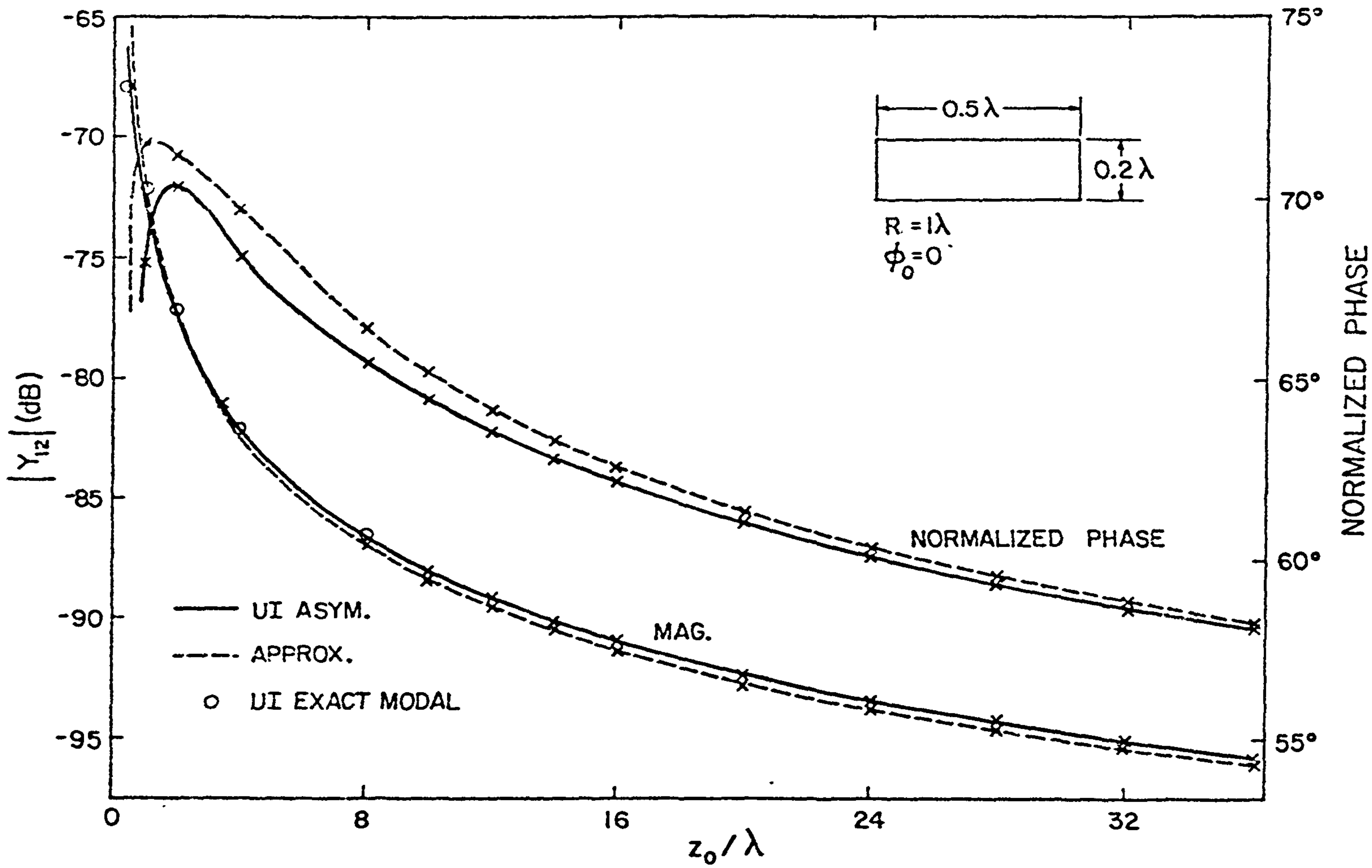


Figure 2. Mutual admittance Y_{12} between two circumferential slots as a function of z_0.

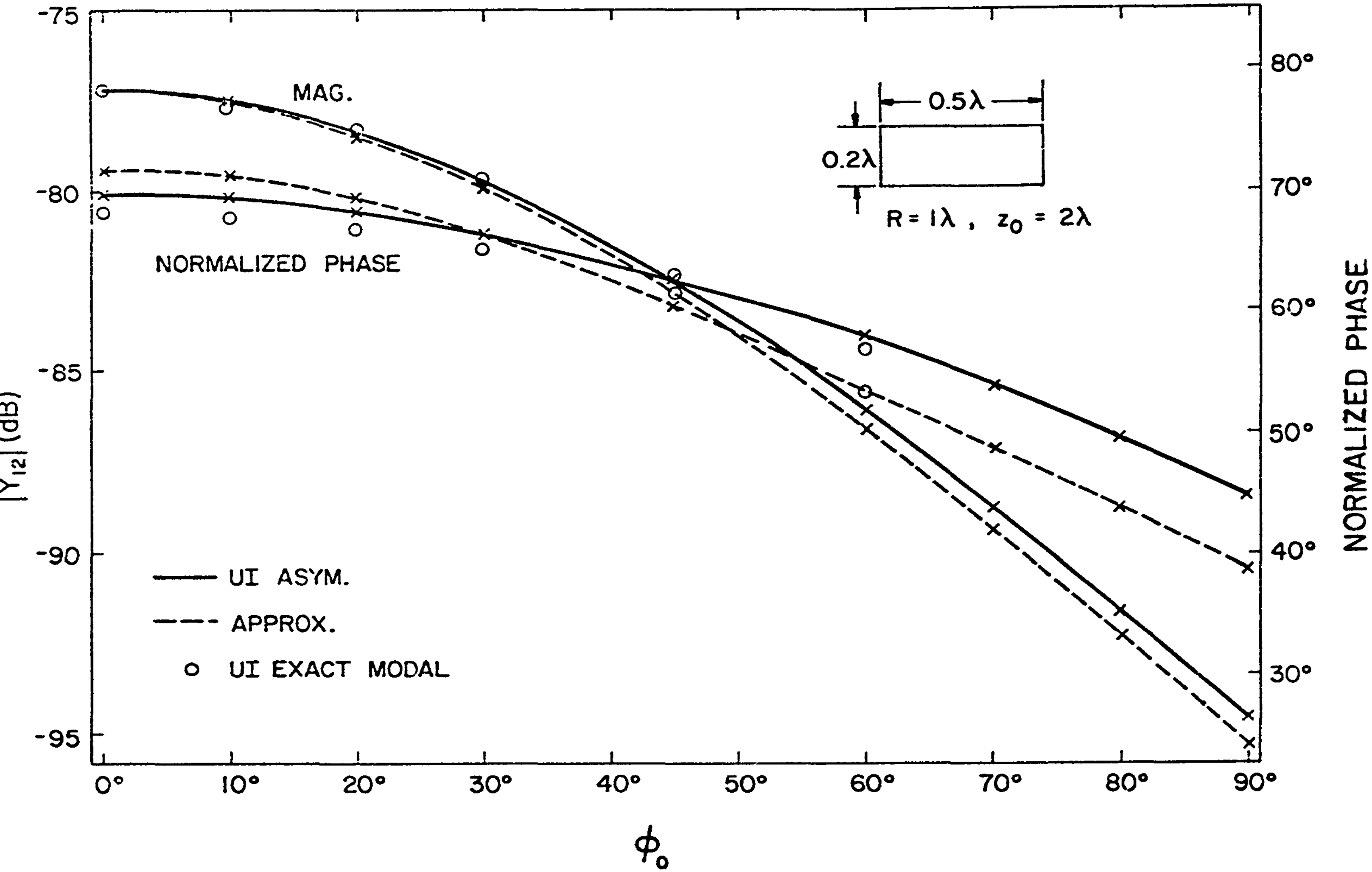


Figure 3. Mutual admittance Y_{12} between two circumferential slots as a function of ϕ_0.

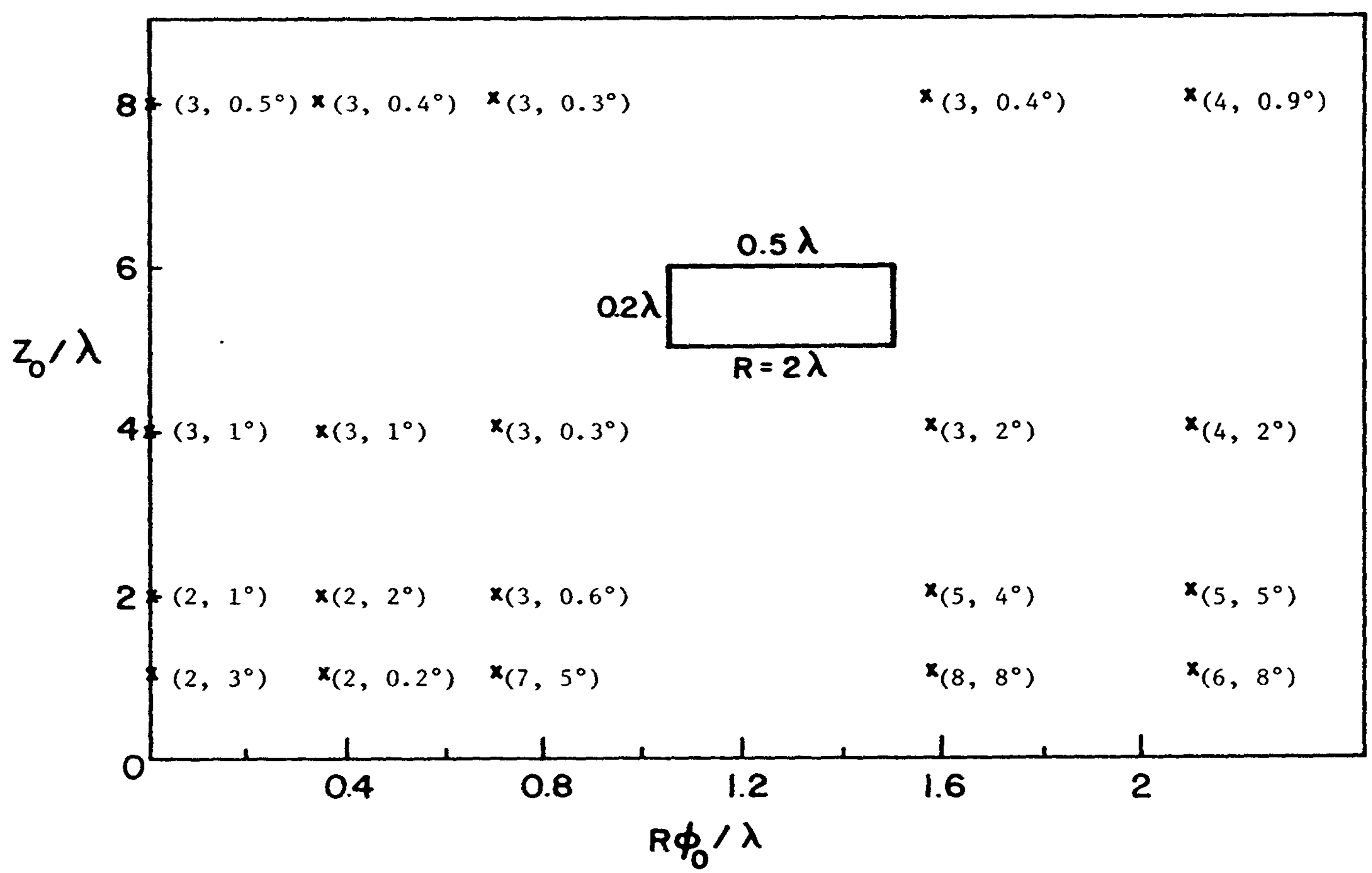


Figure 4. The percentage error in magnitude and absolute error in phase of the approximate formula of Y_{12} of circumferential slots as a function of their relative positions.

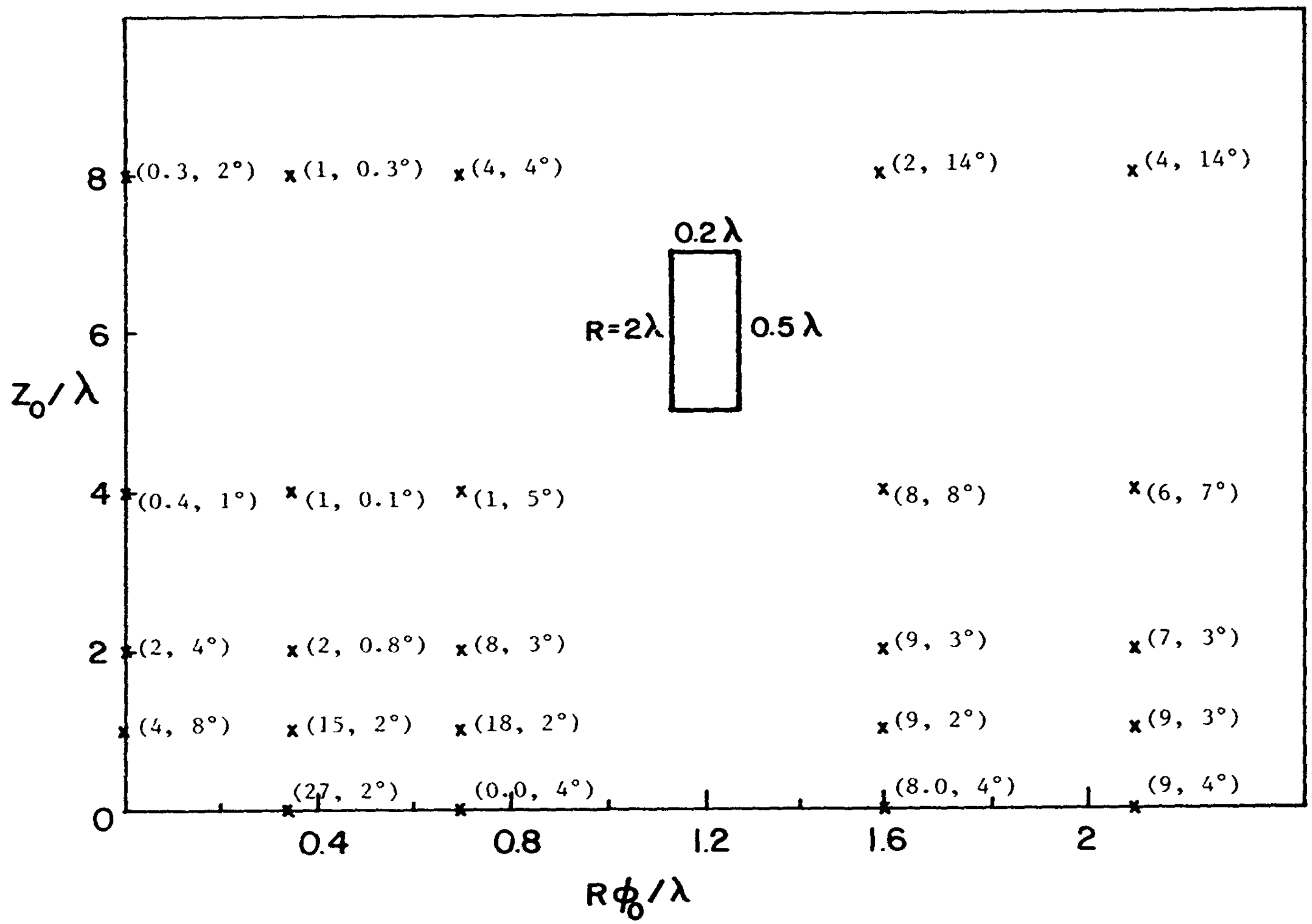


Figure 5. The percentage error in magnitude and absolute error in phase of the approximate formula of Y_{12} of axial slots as a function of their relative positions.

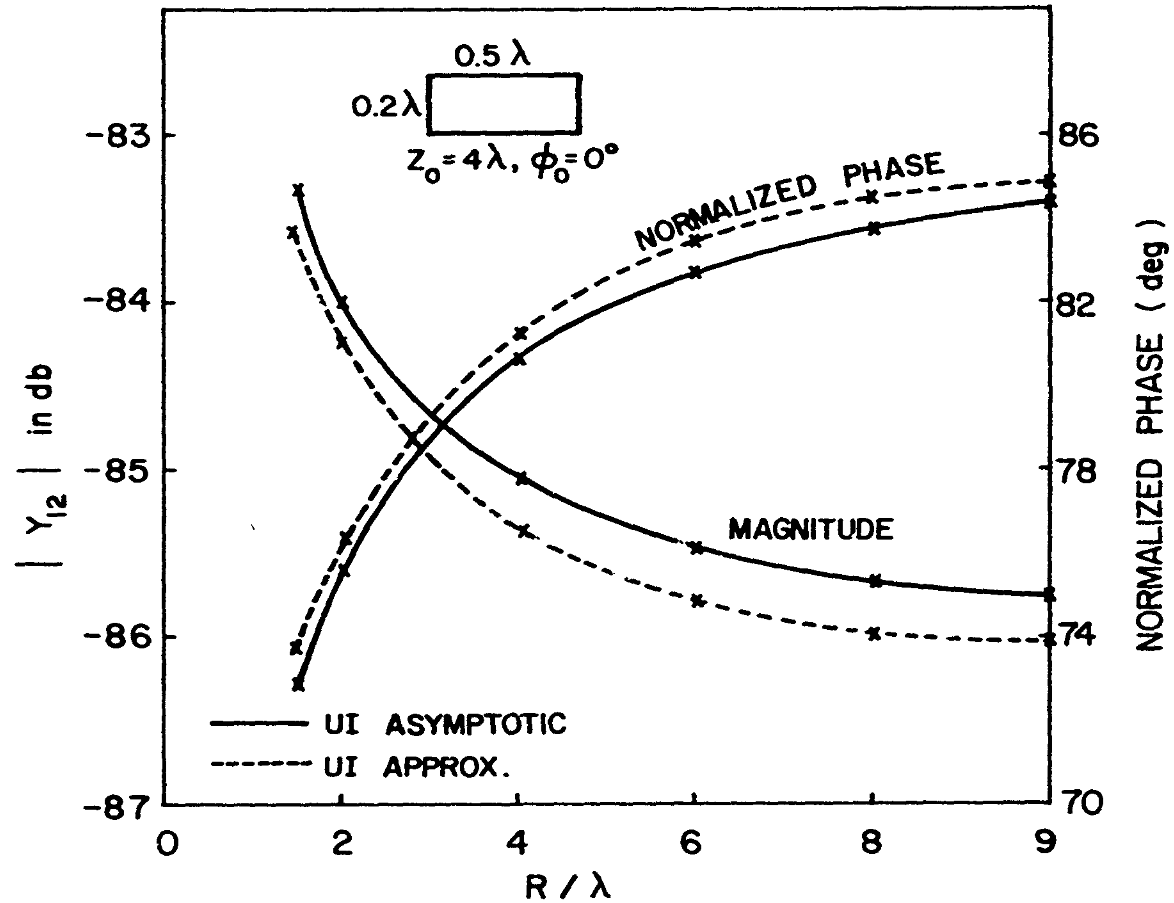
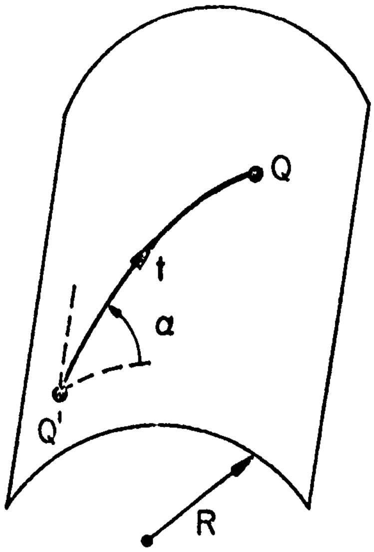
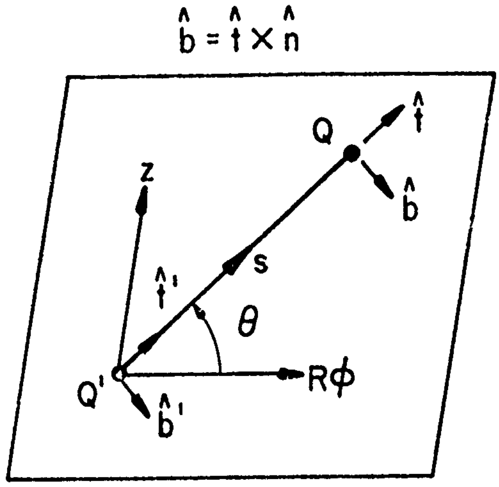


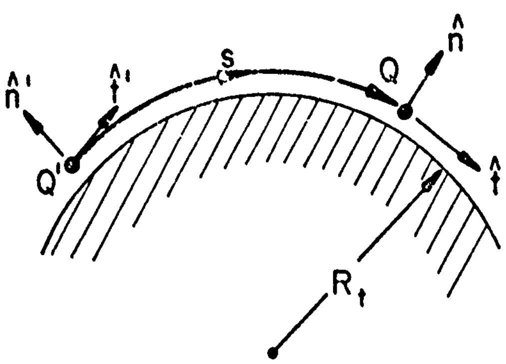
Figure 6. Mutual admittance Y_{12} between two identical circumferential slots as a function of radius R of the cylinder.



(a) 3-D view



(b) Developed cylinder



(c) Cut along θ-direction

Figure 7. A surface ray from source point Q' to observation point Q on a cylinder of radius R.

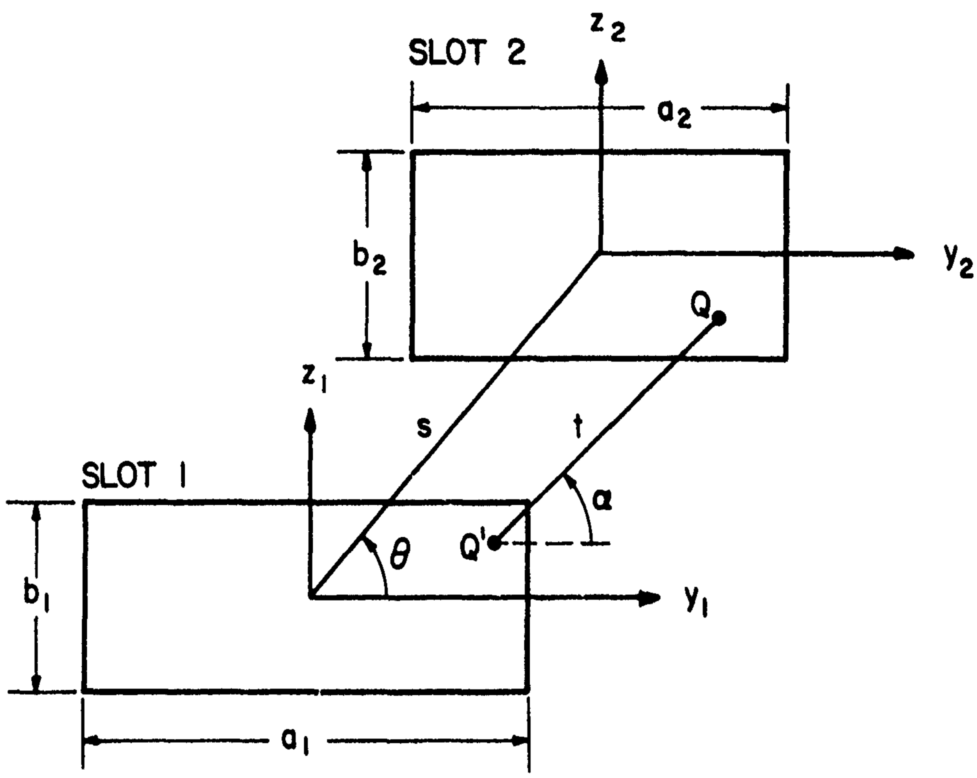


Figure 8. Two circumferential slots on a developed cylinder.

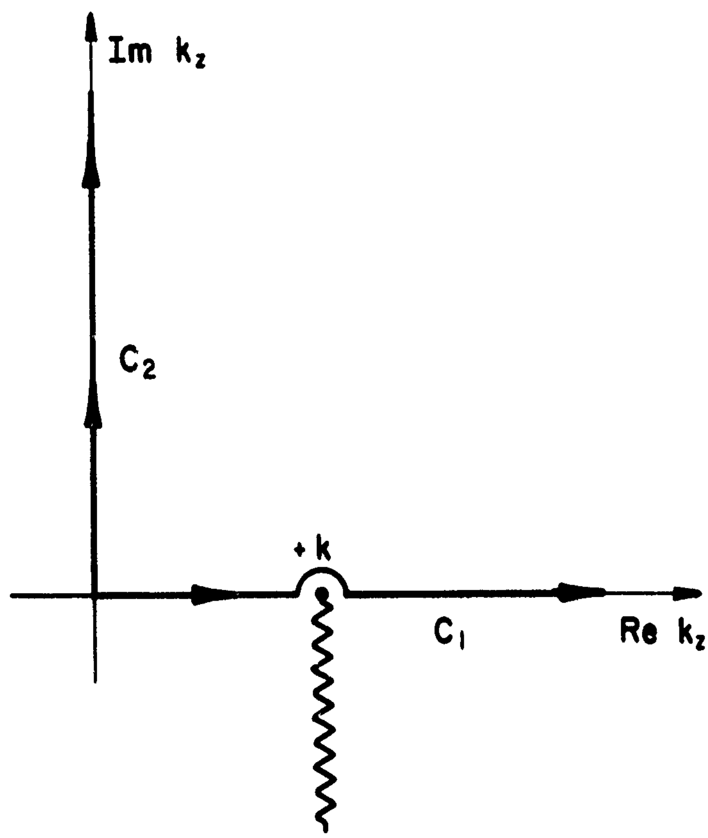


Figure 9. Contours in the complex k_z-plane for the integral in (5.4).

TABLE 1

MUTUAL ADMITTANCE Y_{12} BETWEEN TWO SLOTS

ON A PLANE (E-PLANE COUPLING)

z_0	Exact	Approximate
0.5λ	-64.57 dB -110°	-63.25 dB -108°
1λ	-69.48 78°	-69.58 81°
2λ	-75.13 84°	-75.68 85°
3λ	-78.58 86°	-79.22 87°
4λ	-81.06 87°	-81.72 88°
8λ	-87.05 88°	-87.75 89°

TABLE 2

MUTUAL ADMITTANCE Y_{12} BETWEEN SLOTS ON

A PLANE (H-PLANE COUPLING)

y_0	Exact	Approximate
1λ	-83.41 dB -53°	-85.04 dB -180°
2λ	-96.75 -168°	-97.09 -180°
3λ	-104.00 -172°	-104.13 -180°
4λ	-109.07 -174°	-109.13 -180°
8λ	-121.18 -177°	-121.17 -180°

ATTACHMENT C

MUTUAL ADMITTANCE OF SLOTS ON A CONE:
SOLUTION BY RAY TECHNIQUE*

MUTUAL ADMITTANCE OF SLOTS ON A CONE: SOLUTION BY RAY TECHNIQUE*

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ABSTRACT

An approximate asymptotic Green's function for the surface magnetic field due to a magnetic dipole on a general convex conducting surface is developed. Based largely on the classical work of V. A. Fock and the current GTD recipes, this solution is presented in a form that admits ray interpretations, and can be simply evaluated. We apply the Green's function to calculate the mutual admittance between two slots on a cone. The numerical results are in very good agreement with experiments.

* This work was supported by Naval Air Systems Command under Contract N00019-77-C-0127.

ACKNOWLEDGEMENT

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TABLE OF CONTENTS

1. INTRODUCTION

In the past few years there has been an increasing interest in the study of conformal arrays, where the radiating elements are arranged on a curved conducting surface. A crucial as well as challenging problem in the design of a conformal array is determining the mutual coupling among elements [1], [2]. The present paper addresses itself to one such problem when

(i) the conducting surface is an infinite cone;

(ii) the radiating elements are slots with thin width and a length of about a half wavelength; and

(iii) all the elements are distributed in a region which is away from the cone tip and whose radii of curvature are large in terms of wavelength.

Because of assumption (ii), the aperture field of a slot can be well-approximated by a simple cosine distribution, i.e., the so-called "one-mode approximation." Then it has been established, e.g., p. 53 of [3] or p. 8 of [4], that the calculation of the mutual coupling is reduced to that of a dyadic Green's function due to a magnetic dipole on the same curved conducting surface where the array is at.

The Green's function of a cone can be calculated in the following two ways. Via normal modes involving spherical Bessel functions and associated Legendre functions, it can be expressed exactly in terms of a doubly infinite series [5], [6]. The numerical evaluation of such a series, however, is quite tedious, especially at high frequencies. Thus, up to now, no systematic numerical results have been generated from the series. An alternative way to calculate the Green's function is to employ the surface rays, which would yield a simple approximate solution valid for high frequencies. The general concept of surface rays was introduced by J. B. Keller more than twenty years ago [7] - [9]; however, a uniformly valid formula for fields on the ray has not been developed until recently. Among the several comparable formulas [10] - [12], we chose the one reported in [12] for the present application. The reason for our choice is that, at least for the case of a cylinder, the formula in [12] gives the most accurate numerical results [13].

The organization of this paper is as follows. The formula in [12] for the Green's function applies only to a cylinder. Following the GTD recipe, we generalize it to an arbitrary convex surface with its final solution presented in Section 2, and its derivation in Section 6. In Sections 3 to 5, the Green's function is specialized to a cone and is used to calculate the mutual admittance between two slots on a cone. A conclusion is given in Section 7. The two appendices contain (A) formulas for the Fock functions, and (B) the computer listing for calculating the mutual admittance on a cone.

2. SOLUTION OF THE GREEN'S FUNCTION

Consider a perfectly conducting convex surface Σ (Fig. 1), whose radii of curvature at any point are large in terms of wavelength. At a point Q_1, described by position vector $\vec{r}_1$ on Σ, there is a tangential magnetic dipole source described by a magnetic current density (for exp + $j\omega t$ time convention)

$$\vec{K}(\vec{r}) = \vec{M}\delta(\vec{r} - \vec{r}_1) \tag{2.1}$$

when $\vec{M}$ is the magnetic dipole moment and lies in the tangent plane of Σ. The problem is to determine a high-frequency asymptotic solution of $\vec{H}$ at a general point Q_2 described by position vector $\vec{r}_2$ on Σ. In other words, the dyadic Green's function for the surface magnetic field for points $\vec{r}_1$ and $\vec{r}_2$ is to be found.

Before presenting the solution, let us introduce several definitions and parameters. According to GTD [8], [9], the dominant high-frequency contribution to $\vec{H}(\vec{r}_2)$ is the field on the surface ray from $\vec{r}_1$ to $\vec{r}_2$. The surface ray is a geodesic of Σ. Some of its geometrical properties are described by (Fig. 1)

(i) the arc length $\bar{s}$ which is chosen such that $\bar{s} = 0$ at the source point $\vec{r}_1$ and $\bar{s} = s$ at the observation point $\vec{r}_2$;

(ii) the tangent, normal, and binormal, denoted by $(\hat{t}_n, -\hat{n}_n, -\hat{b}_n)$ at $\vec{r}_n$ where $n = 1,2$; and

(iii) its two radii of curvature $R_t(\bar{s})$, and $R_b(\bar{s})$ of Σ at point $\bar{s}$ in the directions of tangent, and binormal, respectively. (On a general convex surface, both radii are nonnegative.)

From the above parameters, we may calculate the following quantities that are needed for the solution of the Green's function:

(i) The large parameter in our asymptotic expansion of the Green's function is

$$m(\bar{s}) = \left[\frac{1}{2}\, kR_t(\bar{s})\right]^{1/3} , \tag{2.2}$$

which is a function of position along the ray from $\vec{r}_1$ to $\vec{r}_2$.

(ii) A distance parameter from $\vec{r}_1$ to $\vec{r}_2$ is defined by

$$\xi = \int_{\vec{r}_1}^{\vec{r}_2} \frac{k}{2m^2(\bar{s})}\, d\bar{s} . \tag{2.3}$$

For the special case when R_t is not a function of $\bar{s}$ (constant ray curvature), ξ is reduced to $(ks/2m^2)$, a well-known parameter introduced first by Fock [14].

(iii) The ray curvatures at the source and observation points enter in a parameter defined by

$$\tau = \left[\frac{ks}{2m(0)\;m(s)\;\xi}\right]^{1/2} , \tag{2.4}$$

which is positive real for a convex surface, and is reduced to unity for the special case of a constant ray curvature.

(iv) Consider a small pencil of surface rays originating from $\vec{r}_1$ and propagating toward $\vec{r}_2$ (Fig. 1). The angle extended by the pencil at $\vec{r}_1$ is $d\psi_1$, and that at $\vec{r}_2$ is $d\psi_2$. The divergence factor DF of the pencil is defined by

$$DF = \left(\frac{s d\psi_1}{\rho d\psi_2}\right)^{1/2} \tag{2.5}$$

where ρ is the caustic distance of the wavefront at $\vec{r}_2$ and is always positive. For example, if Σ is a sphere and $\vec{r}_1$ is the north pole, DF at point $\vec{r}_2 = (r,\theta,\phi)$ is

$$DF = \left(\frac{\theta}{\sin\theta}\right)^{1/2}$$

which varies from one at the north pole ($\theta = 0$) to infinity at the south pole ($\theta = \pi$) as $\vec{r}_2$ moves along a great circle.

(v) The "mean" radii of curvature between $\vec{r}_1$ and $\vec{r}_2$ are defined by

$$\bar{R}_t = [R_t(0)\ R_t(s)]^{1/2} \tag{2.6a}$$

$$\bar{R}_b = [R_b(0)\ R_b(s)]^{1/2} \quad . \tag{2.6b}$$

Throughout this work, we always assume that Σ is a smooth surface with a slowly varying curvature. Then $(\bar{R}_t,\bar{R}_b)$ represents a sort of average value of radii of curvature along the ray.

Return to the electromagnetic problem in Fig. 1. We assume that $m(\bar{s})$ is large and is slowly varying for all $\bar{s}$ in the range $0 \le \bar{s} \le s$. Then an approximate asymptotic solution for the surface magnetic field at $\vec{r}_2$ due to the dipole source in (2.1) is given by

$$\vec{H}(\vec{r}_2) \approx \vec{M} \cdot (\hat{b}'\hat{b}H_b + \hat{t}'\hat{t}H_t)(DF) \tag{2.7}$$

where

$$H_b = G(s)\left\{\left(1 - \frac{j}{ks}\right) \tau v(\xi) - \left(\frac{1}{ks}\right)^2 \tau^3 u(\xi) + j(\sqrt{2}k\bar{R}_t)^{-2/3} \cdot [\tau v'(\xi) + (\bar{R}_t/\bar{R}_b)\ \tau^3 u'(\xi)]\right\}$$

$$H_t = G(s)\left(\frac{j}{ks}\right)\left[\tau v(\xi) + \left(1 - \frac{2j}{ks}\right)\ \tau^3 u(\xi) + j(\sqrt{2}k\bar{R}_t)^{-2/3}\ \tau^3 u'(\xi)\right]$$

$$G(s) = \frac{k^2 Y}{2\pi j}\,\frac{e^{-jks}}{ks}\ , \quad Y = (120\pi)^{-1} \quad .$$

The Fock functions u and v and their derivatives u' and v' are described in Appendix A. Several remarks about the solution in (2.7) are in order. (i) It is derived in an approximate manner from the classical work of Fock [14] and the recipe of GTD [8], [9], as detailed in Section 6. This solution is certainly not valid when the curvature of the surface Σ is large or rapidly varying. (ii) For the special case that Σ is a planar surface ($R_t = R_b \to \infty$), (2.7) recovers the known <u>exact</u> solution, namely,

$$H_b = G(s)\left[1 - \frac{j}{ks} - \left(\frac{1}{ks}\right)^2\right] \tag{2.8a}$$

$$H_t = G(s)\left(\frac{2j}{ks}\right)\left(1 - \frac{j}{ks}\right) \tag{2.8b}$$

$$DF = 1 \quad . \tag{2.8c}$$

(iii) The solution is valid for any combination of $\vec{r}_1$ and $\vec{r}_2$. In the penumbra region ($\vec{r}_2$ is close to $\vec{r}_1$ and $\xi << 1$), (2.7) is nearly the planar solution in (2.8). In the deep shadow ($\xi << 1$), the residue series representation of the Fock functions in Appendix A may be used, and (2.7) is identified as the creeping-wave contribution. (iv) When Σ is a cylindrical surface, the formula (2.7) has been used to calculate the mutual admittance between two slots on Σ. It has been shown [12], [13] that the numerical results are in excellent agreement with a known exact solution [15] - [17]. (v) Except for the very simple surfaces such as a cylinder, cone, or sphere, no explicit parametric equations can be found for the geodesics. Thus, for a general surface, one may have to rely on numerical techniques for determining the geodesics and the divergent factor, as has been done in [18].

3. GREEN'S FUNCTION OF A CONE

Let us apply the formula (2.7) to the field on an infinite cone, described by the equations (Fig. 2a)

$$x = r \sin \theta_0 \cos \phi \ , \quad y = r \sin \theta_0 \sin \phi \ , \quad z = r \cos \theta_0 \tag{3.1}$$

where θ_0 is the half cone angle ($0 < \theta_0 < \pi/2$). Since the cone is a developable surface, the rays (geodesics) on a developed cone (Fig. 2b) are straight lines. Due to the source at $\vec{r}_1 = (r_1, \theta_0, \phi_1)$, the main contribution of the field at $\vec{r}_2 = (r_2, \theta_0, \phi_2)$ comes from the shortest ray described by

$$r_1 \sin \Omega_1 = r_2 \sin \Omega_2 \quad . \tag{3.2}$$

As the ray propagates away from the source point $\vec{r}_1$, it reaches the highest altitude at M where $\Omega = \pi/2$. After M, the ray travels downward away from the cone tip. The various parameters defined in Section 2 can be simply calculated from the cone geometry [6], [19], and expressed in terms of coordinates (r_1, ϕ_1) and (r_2, ϕ_2). The arclength is

$$s = \left\{ r_1^2 + r_2^2 - 2 r_1 r_2 \cos \left[(\phi_1 - \phi_2) \sin \theta_0 \right] \right\}^{1/2} \quad . \tag{3.3}$$

The angle Ω_1 at $\vec{r}_1$ is

$$\Omega_1 = \sin^{-1} \left\{ \frac{r_2}{s} \sin \left[(\phi_2 - \phi_1) \sin \theta_0 \right] \right\} \quad . \tag{3.4}$$

We choose $|\Omega_1| < \pi/2$ if $r_2^2 < s^2 + r_1^2$, and $|\Omega_1| > \pi/2$ if otherwise. The other parameters are

$$\Omega_2 = \Omega_1 + (\phi_2 - \phi_1) \sin \theta_0 \tag{3.5}$$

$$\bar{R}_t = \frac{\sqrt{r_1 r_2} \tan \theta_0}{\sin \Omega_1 \sin \Omega_2} \quad , \quad \bar{R}_b = \frac{\sqrt{r_1 r_2} \tan \theta_0}{\cos \Omega_1 \cos \Omega_2} \tag{3.6}$$

$$\xi = \left(\frac{1}{2} kr_1 \sin \Omega_1 \sin \theta_0\right)^{1/3} |\phi_2 - \phi_1| \cos^{2/3} \theta_0 \tag{3.7}$$

$$\tau = (ks/\xi)^{1/2} \left(2k^2 r_1 r_2\right)^{-1/6} (\sin \Omega_1 \sin \Omega_2 \cot \theta_0)^{1/3} \tag{3.8}$$

$$DF = 1 \quad . \tag{3.9}$$

When the above parameters in (3.3) through (3.9) are substituted into (2.7), we obtain an approximate solution for the surface field on a cone due to a direct surface ray contribution. Let us consider a special observation point $\vec{r}_2$ such that

$$ks >> 1 \text{ , } \Omega_1 \text{ and } \Omega_2 \text{ are not close to } \pi/2 \quad . \tag{3.10}$$

Then the two components of the field in (2.7) are reduced to, after making used of the residue series representations for the Fock functions (Appendix A) and keeping only the leading terms,

$$H_b \sim \frac{k^2 (\sin \Omega_1 \sin \Omega_2 \cot \theta_0)^{1/3}}{1528 \left(k^2 r_1 r_2\right)^{1/6} (ks)^{1/2}} \exp \left[-0.88\xi - j\left(\frac{5\pi}{12} + 0.51\xi + ks\right)\right] \tag{3.11a}$$

$$H_t \sim O[(ks)^{-3/2}] \tag{3.11b}$$

which agrees with the rigorous asymptotic solution given in Eqs. (50) and (53) of [6]. (In making the comparison, note the corresponding notations used in [6] and here: $-i \to j$, $\theta_c \to \theta_0$, $L_1 \to s$, $r_> \to r_1$, $r_< \to r_2$, $\beta_{s>} \to \pi/2 - \Omega_1$, and $\bar{q}_1 \to |t_1'|$.) We emphasize that the result in (3.11) or that in [6] is valid only under the conditions in (3.10). For an arbitrarily located observation point, (2.7) should be used.

Two final remarks about the formula in (2.7) are in order. (i) For a given source and observation point, there are infinitely many rays (geodesics) passing through them. The contribution from each ray may be calculated from (2.7), and the final field solution is the superposition of all ray contributions. In most practical problems (all the numerical computations presented in this paper), only the ray with the shortest arclength gives the significant contribution to the field solution, whereas all other rays may be ignored. (ii) Depending on the polarization and the distances of the source and observation points from the cone tip, there may be another significant contribution to the field from the diffraction at the tip. In such a case, the total field at any point contains two dominant contributions: one from the direct ray according to formula (2.7), and one from the tip-diffracted ray. More about the latter will be given in Section 4.

4. MUTUAL ADMITTANCE ON A CONE

On the surface of a cone, let us consider two arbitrarily oriented slots. Under the assumption that the dimensions of the slots are relatively small compared with the radii of curvature of the cone surface, the shapes of slots are taken to be rectangular on a developed cone. Note that, depending on the exact manner in which the feeding waveguide is fitted into the cone surface, the shape of a slot mapped on a developed cone may be quite irregular. Our assumption of rectangular shapes represents a good approximation for practical cases; at the same time, it simplifies the subsequent calculations.

Referring to Fig. 3, we describe the dimensions and the positions of the two slots by

$$(a_n, b_n) \text{ and } [c_n, (n-1)\,\phi_0, \omega_n], \quad n = 1,2 \quad .$$

Thus, the radial separation of the two slots is $(c_2 - c_1)$ and the angular separation is ϕ_0. The angle ω_n measures the deviation of the longitudinal direction of slot n from the radial direction of the cone. If $\omega_n = 0$, slot n is radial; if $\omega_n = \pi/2$, slot n is circumferential. The mutual admittance Y_{12} between the two slots is defined as follows. Throughout this work we always assume that

(i) the slots are thin, and (4.1a)

(ii) their length is roughly a half-wavelength. (4.1b)

Then the aperture field in each slot can be adequately approximated by a simple cosine distribution, which is the so-called "one-mode" approximation. The aperture field of slot 1 under the "one-mode" approximation is given by

$$\vec{E} = V_1 \vec{e}_1 \ , \ \vec{H} = I_1 \vec{h}_1 \tag{4.2a}$$

where

$$\vec{e}_1 = \hat{z}_1 \sqrt{\frac{2}{a_1 b_1}} \cos \frac{\pi}{a_1} y_1 \quad , \quad \vec{h}_1 = \hat{x}_1 x \vec{e}_1 \quad . \tag{4.2b}$$

Here (y_1, z_1) are the local rectangular coordinates, with the origin at the center of slot 1 and y_1-axis parallel to the longer dimension of the slot. (V_1, I_1) are respectively the modal (voltage, current) of slot 1. The mutual admittance Y_{12} is defined by

$$Y_{12} = Y_{21} = \frac{I_{21}}{V_1} \tag{4.3}$$

where I_{21} is the induced current in slot 2 when slot 1 is excited by a voltage V_1 and slot 2 is short-circuited. An alternative expression for Y_{12} is

$$Y_{12} = \frac{1}{V_1 V_2} \iint_{A_2} \vec{E}_2 \times \vec{H}_1 \cdot d\vec{s}_2 = \frac{-1}{V_1 V_2} \iint_{A_2} (\vec{E}_2 \cdot \hat{z}_2)(\vec{H}_1 \cdot \hat{y}_2)\, dy_2\, dz_2 \tag{4.4}$$

where

A_2 = aperture of slot 2,

$\vec{H}_1$ = magnetic field when slot 1 is excited with voltage V_1, and slot 2 is covered by a perfect conductor,

$\vec{E}_2$ = electric field when slot 2 is excited with voltage V_2, and slot 1 is covered by a perfect conductor.

Because $\vec{H}_1 = I_{21}\vec{h}_2$ and $\vec{E}_2 = V_2\vec{e}_2$, it is a simple matter to verify that (4.3) and (4.4) are equivalent.

At high frequencies, $\vec{H}_1$ in (4.4) has two dominant contributions: one from the direct rays going from slot 1 to slot 2, and the other from the rays diffracted at the tip of the cone, namely,

$$\vec{H}_1 \sim \vec{H}_1^d + \vec{H}_1^t \quad . \tag{4.5}$$

Accordingly, Y_{12} also has two parts

$$Y_{12} \sim Y_{12}^d + Y_{12}^t \quad . \tag{4.6}$$

Let us concentrate on Y_{12}^d first. Making use of the Green's function in (2.7) and the aperture distribution in (4.2), Y_{12}^d may be explicitly written as

$$Y_{12}^d = \frac{-2k^2}{(a_1b_1a_2b_2)^{1/2}} \int_{-a_1/2}^{a_1/2} dy_1 \int_{-b_1/2}^{b_1/2} dz_1 \int_{-a_2/2}^{a_2/2} dy_2 \int_{-b_2/2}^{b_2/2} dz_2$$

$$\times \left(\cos \frac{\pi}{a_1} y_1\right) \left(\cos \frac{\pi}{a_2} y_2\right) g(y_1z_1;y_2z_2) \tag{4.7a}$$

where

$$g(y_1,z_1;y_2,z_2) = H_b \cos \omega_3 \cos \omega_4 + H_t \sin \omega_3 \sin \omega_4 \quad . \tag{4.7b}$$

The Green's function components (H_b,H_t) are given in (2.7), and angles (ω_3,ω_4) are shown in Fig. 3. In evaluating the integrals in (4.7a), for two given points (y_1,z_1) and (y_2,z_2), we must calculate some geometrical parameters appearing in H_b and H_t. Those calculations lead to the following results

$$r_n = \left[c_n^2 + y_n^2 + z_n^2 - 2c_n \sqrt{y_n^2 + z_n^2} \cos(\omega_n - \omega_{n+4})\right]^{1/2} \tag{4.8a}$$

$$\phi_n = (\sin \theta_0)^{-1} \sin^{-1} \left[\sqrt{y_n^2 + z_n^2}\, r_n^{-1} \sin(\omega_n - \omega_{n+4})\right] \tag{4.8b}$$

$$\omega_{n+4} = \tan^{-1}(z_n/y_n) \tag{4.8c}$$

$$\omega_{n+2} = \Omega_n + (\pi/2) - \omega_n - \phi_n \sin \theta_0 + (n-1)\,\phi_0 \sin \theta_0 \tag{4.8d}$$

where n = 1 and 2. We evaluate the integrals in (4.7a) numerically with the aid of a computer.

Next let us consider Y_{12}^t, the part of mutual admittance due to the rays diffracted at the cone tip. For the special case of circumferential slots, an approximate expression of Y_{12}^t is given in [16], which reads

$$Y_{12}^t \approx T, \text{ if } \omega_1 = \omega_2 = \pi/2 \quad , \tag{4.9a}$$

where

$$T = \sigma_0 \frac{(a_1 b_1 a_2 b_2)^{1/2}}{30\pi^4 c_1 c_2 \sin \theta_0} \left(\frac{\tan \theta_0}{2\pi}\right)^{1/2} \frac{\sin (kb_1/2) \sin (kb_2/2)}{(kb_1/2)(kb_2/2)}$$
$$\cdot \exp j\left(\frac{\pi}{4} - kc_1 - kc_2\right) \quad . \tag{4.9b}$$

Here σ_0 is the zeroth-order tip diffraction coefficient and is a function of the half cone angle θ_0. A numerical table of σ_0 for several typical values of θ_0 is given in [16]. We have fitted those values by a simple expression, viz.,

$$\sigma_0 = A \exp jB \quad , \tag{4.10}$$

where

$$A = 1.3057\theta_0^{-1} - 1.755 + 2.772\theta_0 - 1.459\theta_0^2$$
$$B = 2.7195 + 1.4608\theta_0 - 1.1295\theta_0^2 + 0.6566\theta_0^3$$

Both θ and B are in radians.

As may be seen from Fig. 4, the numerical values of σ_0 calculated from (4.10) are in excellent agreement with those tabulated in [16]. For the special case of axial slots, Y_{12}^t according to [16] is approximately

$$Y_{12}^t \approx 0 \text{ , if } \omega_1 = \omega_2 = 0 \quad . \tag{4.11}$$

In the present paper, we are interested in the general case that the two slots have arbitrary orientations. Before a more exact solution can be found, we will use the following formula

$$Y_{12}^{t} \approx T \sin \omega_1 \sin \omega_2 \tag{4.12}$$

which matches the two extreme cases in (4.9) and (4.11), and interpolates the in-between cases by regarding each slot as a thin magnetic dipole.

5. NUMERICAL RESULTS OF MUTUAL ADMITTANCE

The final solutions for Y_{12} (total mutual admittance) and Y_{12}^d (partial mutual admittance) are given in (4.6), (4.7), (4.12), and (4.9b). For a given geometry of the slots and cone, the two surface integrals in (4.7a) are evaluated numerically by choosing an integration grid roughly equal to $0.05\lambda \times 0.05\lambda$. Thus, for two typical $0.5\lambda \times 0.2\lambda$ slots, the integrals are replaced by a summation of 1600 terms. Fortunately, the integrand is simple enough that each Y_{12} calculation takes about one second on the CDC Cyber 170 Series Computer Systems. We have analyzed a large number of cases for Y_{12}. Typical results are summarized below.

Unless specified otherwise, all numerical computations are based on two identical slots with

$$\text{slot length} = 0.5\lambda \text{ , width} = 0.2\lambda \quad . \tag{5.1}$$

The other parameters are the half cone angle θ_0, the slot orientations (ω_1,ω_2), the distances from slot centers to the cone tip (c_1,c_2), and the slot angular separation $\phi_0 = \phi_2 - \phi_1$ (Fig. 3).

(A) <u>"Equivalent" cylinder</u>. It has been conjectured in [16] that, in calculating Y_{12}^d (the contribution from the direct rays) approximately, the cone may be replaced by an "equivalent" cylinder with parameters (Fig. 5)

$$z_0 = c_1 - c_2 \text{ , } \phi_0 = \phi_0 \text{ , } R = \frac{1}{2}(c_1 + c_2)\sin\theta_0 \quad . \tag{5.2}$$

This conjecture can be now quantitatively checked out. In Table I we compare Y_{12}^d on a cone with $\theta_0 = 15°$ or $30°$ calculated from (4.7), and Y_{12} on a cylinder calculated by a similar GTD solution reported in [12]. All values of Y_{12} (or Y_{12}^d) are listed in (DB = 20 $\log_{10} |Y_{12}|$, phase in degree) format. For the cone with the smaller angle ($\theta_0 = 15°$), the "equivalent" cylinder method gives a good approximation for Y_{12}^d. For the cases listed in Table I, the magnitude error of Y_{12}^d is within 0.5 dB (6 percent) and phase error within 15°. For the cone with the larger angle (θ_0 = 30°), however, the "equivalent" cylinder method is not very accurate with magnitude, and phase errors as large as 2.5 dB (33 percent), and 56°, respectively.

(B) Comparison with experiments. A set of experimental data on the mutual coupling between two X-band open-ended waveguides (0.9" × 0.4") on a cone was reported in [16]. As a function of frequency, measurements were done on the coupling coefficient S_{12}, which is related to Y_{12} through the relation

$$S_{12} = \frac{-Y_g Y_{11}}{(Y_g + Y_{11})^2 - Y_{12}^2} \quad . \tag{5.3}$$

Here $Y_g = (120\pi)^{-1}[k^2 - (\pi/a)^2]^{1/2}$ is the admittance of the TE_{10} mode in the feed waveguide. Y_{11} is the self-admittance of a slot on the cone. In the present calculations, we use, instead, the Y_{11} on an "equivalent" cylinder which is calculated by the exact modal solution described in [16] (for example, Y_{11}/Y_g = 0.8178 + j0.3886 at 8.5 GHz, and 0.8591 + j0.3828 at 9 GHz). Since Y_{11} is least sensitive to the geometry, the approximation of a cone by a cylinder should not introduce any significant error in S_{12}. In Figs. 6 and 7, three sets of data are presented: (i) the experimental data; (ii) the theoretical results from the present analysis in which the calculation of Y_{12}^d is based on a cone, e.g., Equation (4.7); (iii) the theoretical results from [16] in which Y_{12}^d is calculated from the exact modal solution of an "equivalent" cylinder. Several observations can be made. (a) Both theoretical results are in good agreement with the experimental data (with the present result being slightly better). As explained in (A), the "equivalent" cylinder method works because the cone angles ($\theta_0 \sim 10°$) are small. (b) The peaks and valleys are caused by the interference between Y_{12}^d and Y_{12}^t, which are of comparable magnitudes due to the large angular separations (60.8° and 80°). (c) There exists a slight shift in frequency ($\Delta f/f \approx 3$ percent) between the theoretical and experimental valleys in Fig. 6. We speculate that this may be due to a slight phase inaccuracy in Y_{12}^t. As a final remark, it has been found experimentally (private communication from G. E. Stewart and K. E. Golden of Aerospace Corporation) that the Y_{12}^t contribution is sensitive to the exact shape of the cone tip. When the tip is not extremely sharp, the peaks and valleys in Figs. 6 and 7 become much less predominant.

(C) Mutual admittances of circumferential slots. In Figs. 8 to 10, Y_{12} and Y_{12}^d for two circumferential slots are displayed as functions of angular separation ϕ_0 and the radial separation $(c_1 - c_2)$. We note that the effect of Y_{12}^t can modify the curves of Y_{12} in several different ways. When the slots are at the same latitude (Fig. 8), the direct coupling is weak. Thus, tip contribution is noticeable even at a small angular separation. As the radial separation is increased (Fig. 9), the tip contribution is almost negligible for $\phi_0 < 65°$. When the two slots are widely separated in the radial direction with one slot near the tip (Fig. 10), the tip contribution gets stronger, and the direct contribution becomes insensitive to ϕ_0. Hence, the oscillation on the Y_{12} curve has a much larger period. In fact, there is only a half "cycle" in the range $0 < \phi_0 < 90°$, and Y_{12} appears to be shifted from Y_{12}^d by a fixed amount.

(D) Effect of slot orientation on mutual admittance. Consider two slots separated by 1 λ along the radial direction. The magnitude of Y_{12} as functions of the slot orientation angles ω_1 and ω_2 is plotted in Fig. 11. As expected, the maximum value (-73 dB) occurs when both slots are circumferential ($\omega_1 = \omega_2 = 90°$). This value is above 14 dB higher than that when both slots are radial ($\omega_1 = \omega_2 = 0$). The minimum value (-113 dB) of Y_{12} occurs when the top slot is radial and the bottom one is circumferential. This result confirms a common belief that the mutual coupling between two orthogonal slots is generallv negligible.

6. DERIVATION OF THE GREEN'S FUNCTION

We will now describe briefly the derivation for the Green's function for the general convex surface given in (2.7).

Our starting point is the corresponding Green's function for a cylinder. For the latter problem, several versions of the asymptotic solutions [10] - [12] exist. All the versions are of similar nature, and contain some approximations that have not yet been fully justified. For the cylinder problem, both [11] and [12] give excellent numerical results (with [12] being slightly better as demonstrated in [13]). We quote below the Green's function for a cylinder reported in [12], which again can be written in the form of (2.7) with DF = 1 and

$$H_b = G(s)\left\{\left(1 - \frac{j}{ks}\right) v(\xi) - \left(\frac{1}{ks}\right)^2 u(\xi) + j(\sqrt{2}\, kR_t)^{-2/3} \cdot [v'(\xi) + (R_t/R_b)\, u'(\xi)]\right\} \tag{6.1a}$$

$$H_t = G(s)\left(\frac{j}{ks}\right)\left[v(\xi) + \left(1 - \frac{2j}{ks}\right) u(\xi) + j(\sqrt{2}\, kR_t)\, u'(\xi)\right] \tag{6.1b}$$

where

$$R_t = R/\cos^2\theta\ ,\quad R_b = R/\sin^2\theta$$

$$\xi = ks(\cos^4\theta/2k^2R^2)^{1/3}$$

R = radius of cylinder

θ = angle between the ray and the ϕ-direction.

Since θ is a constant along a ray (geodesic), so are the three parameters R_t, R_b, and ξ appearing in (6.1). The formula (6.1) is basically derived from Fock's classical solution for vector potentials of a sphere [14], but contains a modification, namely, the last term in (6.1a)

$$G(s)\ j(\sqrt{2}\, kR_t)^{-2/3}(R_t/R_b)\ u'(\xi) \tag{6.2}$$

was added to the Fock's solution in an arbitrary manner. Note that this additional term introduces a very small contribution for a sphere, or for a cylinder as long as θ is not close to $\pi/2$. For $\theta = \pi/2$ on a cylinder, H_b in (6.1a) becomes (see Eq. (2.10) of [12])

$$H_b = [H_b]_{pl} + \frac{3}{8}\sqrt{\frac{1}{2\pi}}\, k^2 Y e^{-j3\pi/4}\, \frac{1}{kR}\, \frac{e^{-jks}}{\sqrt{ks}} \quad . \tag{6.3}$$

Here $[H_b]_{pl}$ is the corresponding solution on an infinite plane and it decays as $(ks)^{-1}$ for large ks. The second term in (6.3) comes from the additional term in (6.2), and is the dominant contribution at large ks because it decays as $(ks)^{-1/2}$. Recently, J. Boersma (private communication) has shown that (6.3) is in exact agreement with a rigorous asymptotic solution for the cylinder. Thus, the additional term in (6.3) is justified for $\theta = \pi/2$ where its contribution is most significant.

Now, let us generalize (6.1) to a general convex surface sketched in Fig. 1 following the GTD recipe [8], [9], [18]: (i) The divergence factor in (2.5) is introduced from the consideration of energy conservation. (ii) The generalized ξ in (2.3) and τ in (2.4) are based on the rigorous solutions of two-dimensional canonical problems. The only remaining problem is the generalization of R_t and R_b in (6.1). We note that the above GTD recipe is valid only if R_t and R_b are slowly varying along a ray. Thus, any sort of average values of R_t and R_b should give approximately the same result. We choose the geometrical mean in (2.6) for its simplicity and symmetry between the source and observation points. From these considerations, we obtain the generalized solution (2.7) from (6.1).

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TABLE I.

Y_{12}^{d} ON CONE AND Y_{12} ON CYLINDER*

ϕ_o	z_o/λ	Cone $\theta_o = 15°$	Cylinder	Cone $\theta_o = 30°$
30°	0	- 91.35dB 155°	- 91.47 153°	- 90.99 160°
60°	0	-110.97 116°	-111.28 101°	-108.78 151°
0	1	- 73.35 73°	- 73.64 73°	- 73.49 74°
0	4	- 83.89 77°	- 84.30 75°	- 84.06 75°
30°	1	- 86.52 - 74°	- 86.59 - 76°	- 86.60 -70°
30°	4	- 86.95 24°	- 87.14 20°	- 86.83 35°
60°	1	-104.64 -35°	-104.17 -42°	-106.28 -13°
60°	4	- 94.41 -118°	- 94.64 -133°	- 94.12 -77°

* The parameters are $\omega_1 = \omega_2 = 90°$, $R = 2\lambda$, $a = 0.5\lambda$, and $b = 0.2\lambda$.

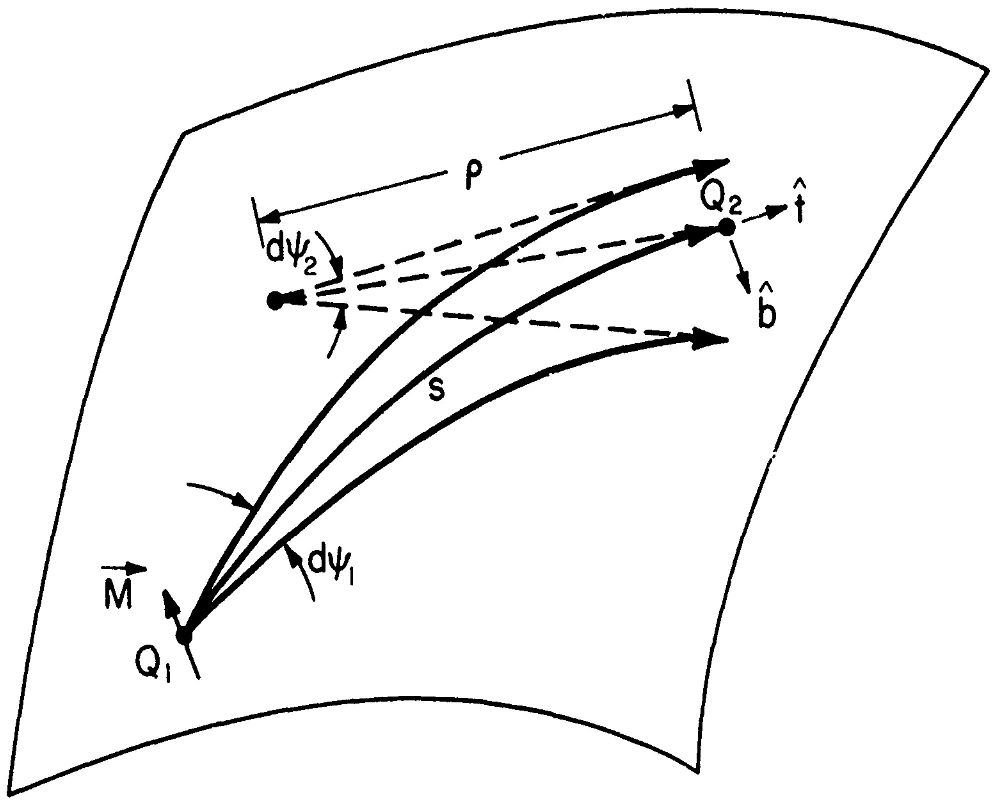


Fig. 1. A surface ray pencil originating from the magnetic dipole source at Q_1. The central ray of the pencil passes through the observation point Q_2. The angle extended by the pencil is $d\psi_1$ at Q_1 and $d\psi_2$ at Q_2.

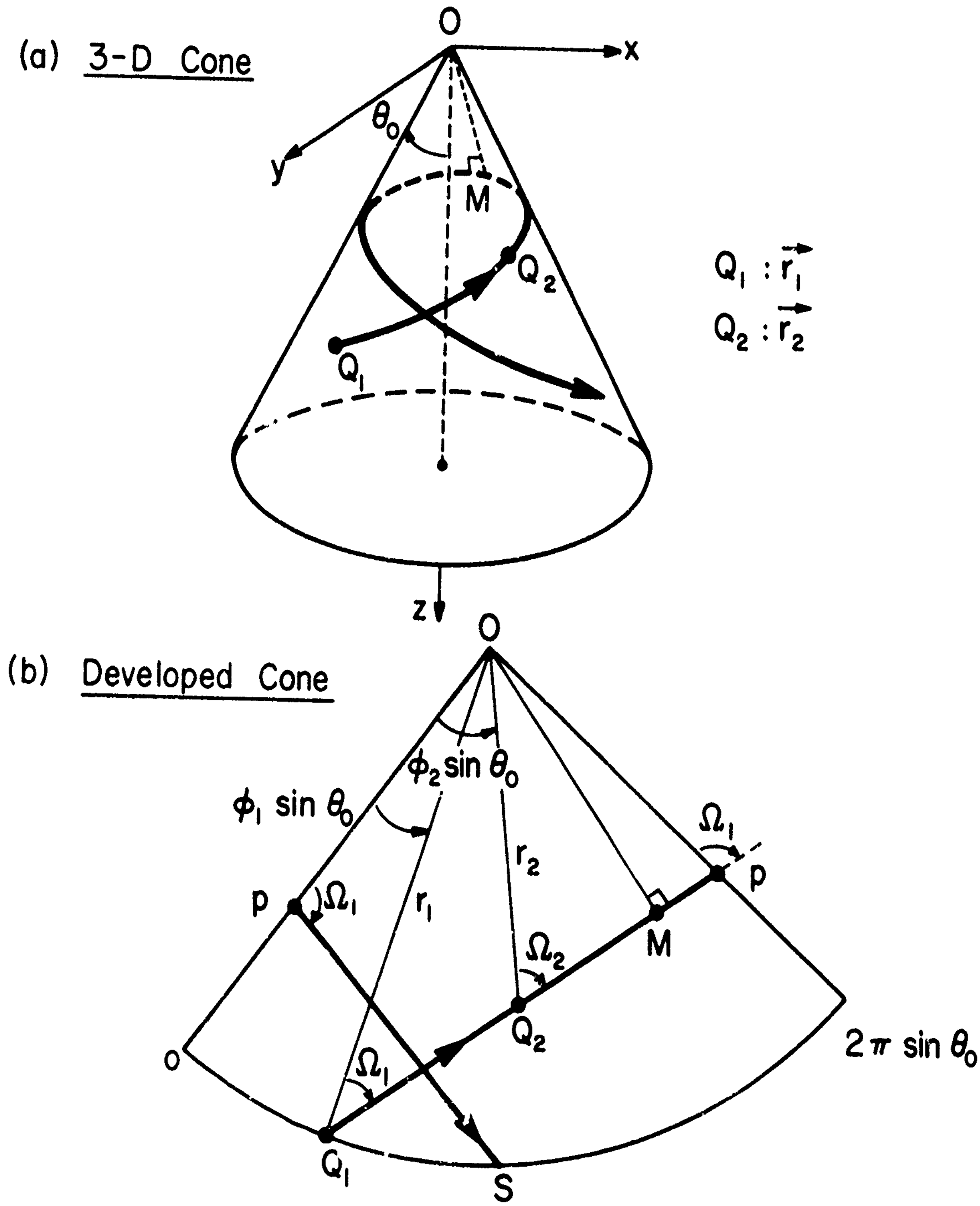


Fig. 2. A surface ray from source point Q_1 to observation point Q_2 on a cone with half cone angle θ_0.

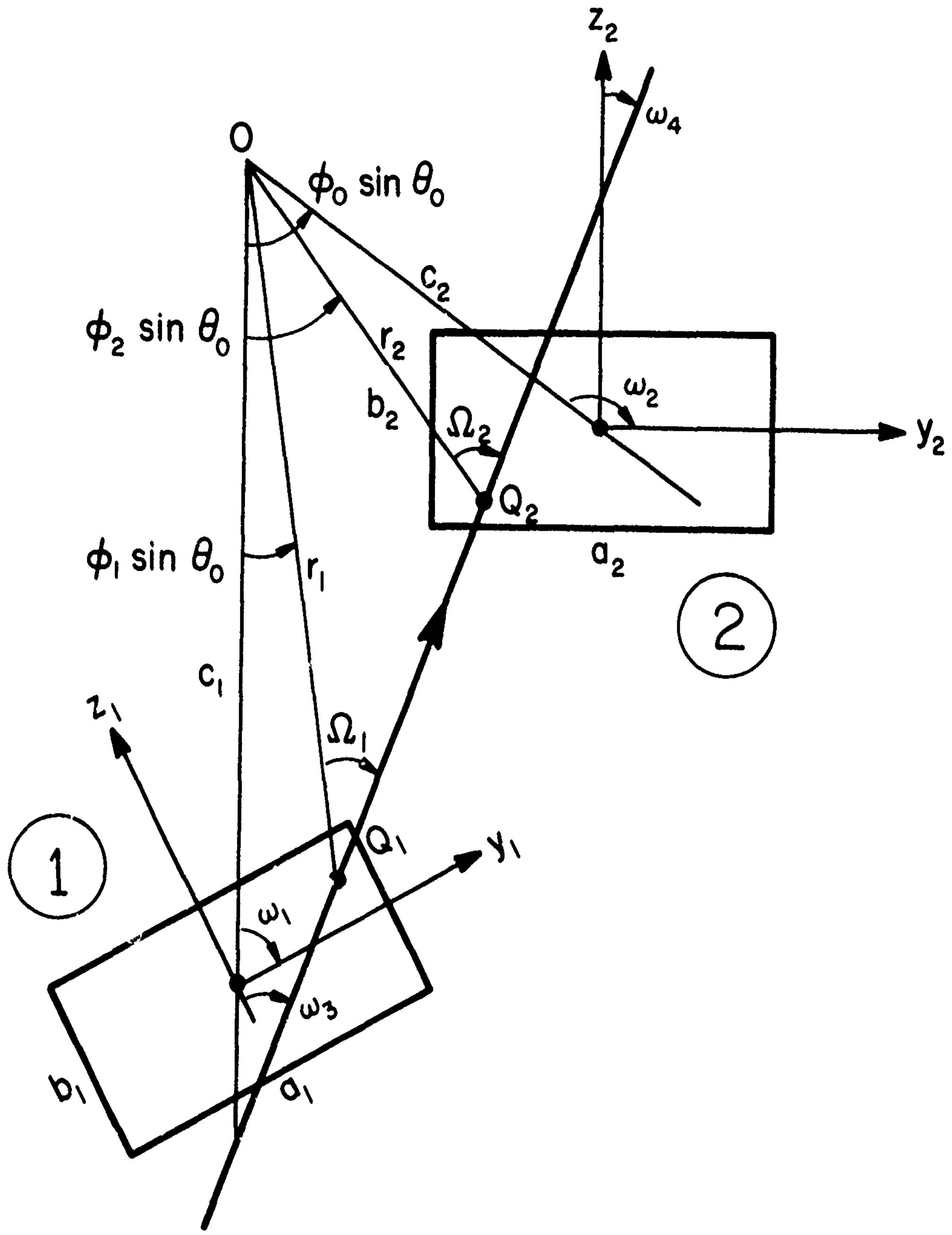


Fig. 3. Two rectangular slots on a developed cone.

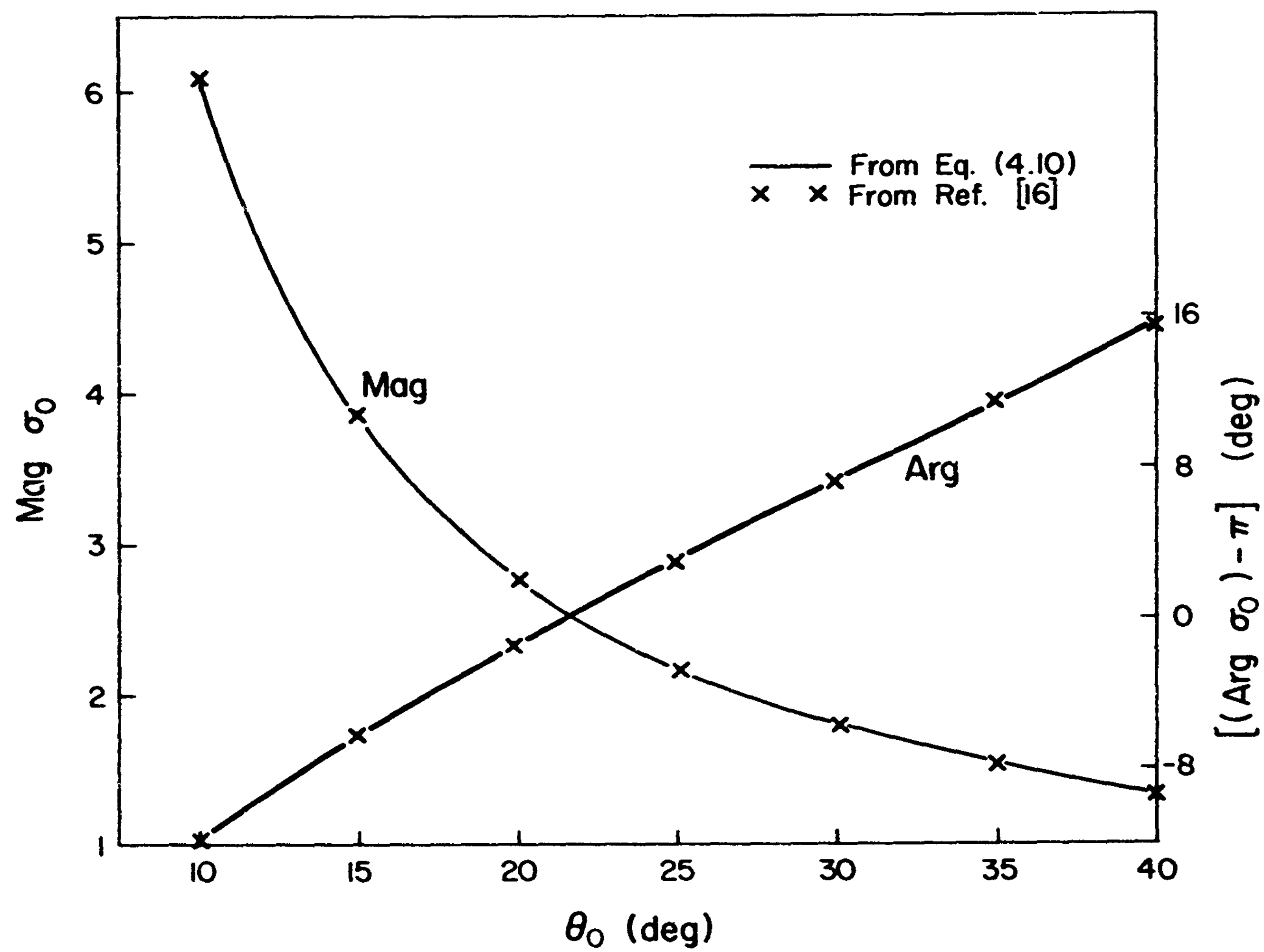


Fig. 4. Tip diffraction coefficient σ_0 of a cone which appeared in (4.9b).

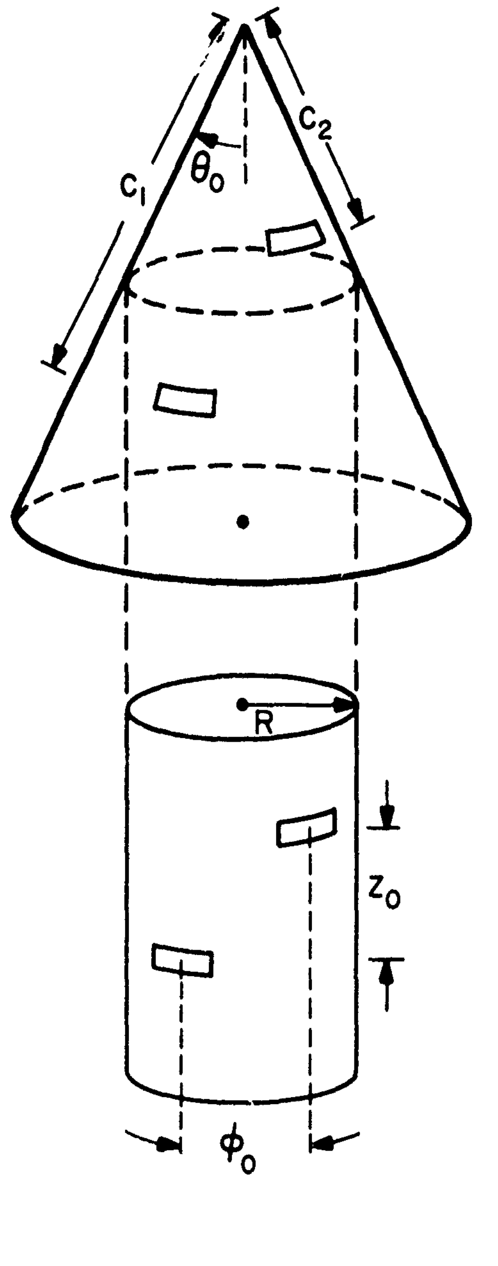


Fig. 5. To calculate Y^d_{12} of a cone approximately, the cone may be locally replaced by an "equivalent" cylinder.

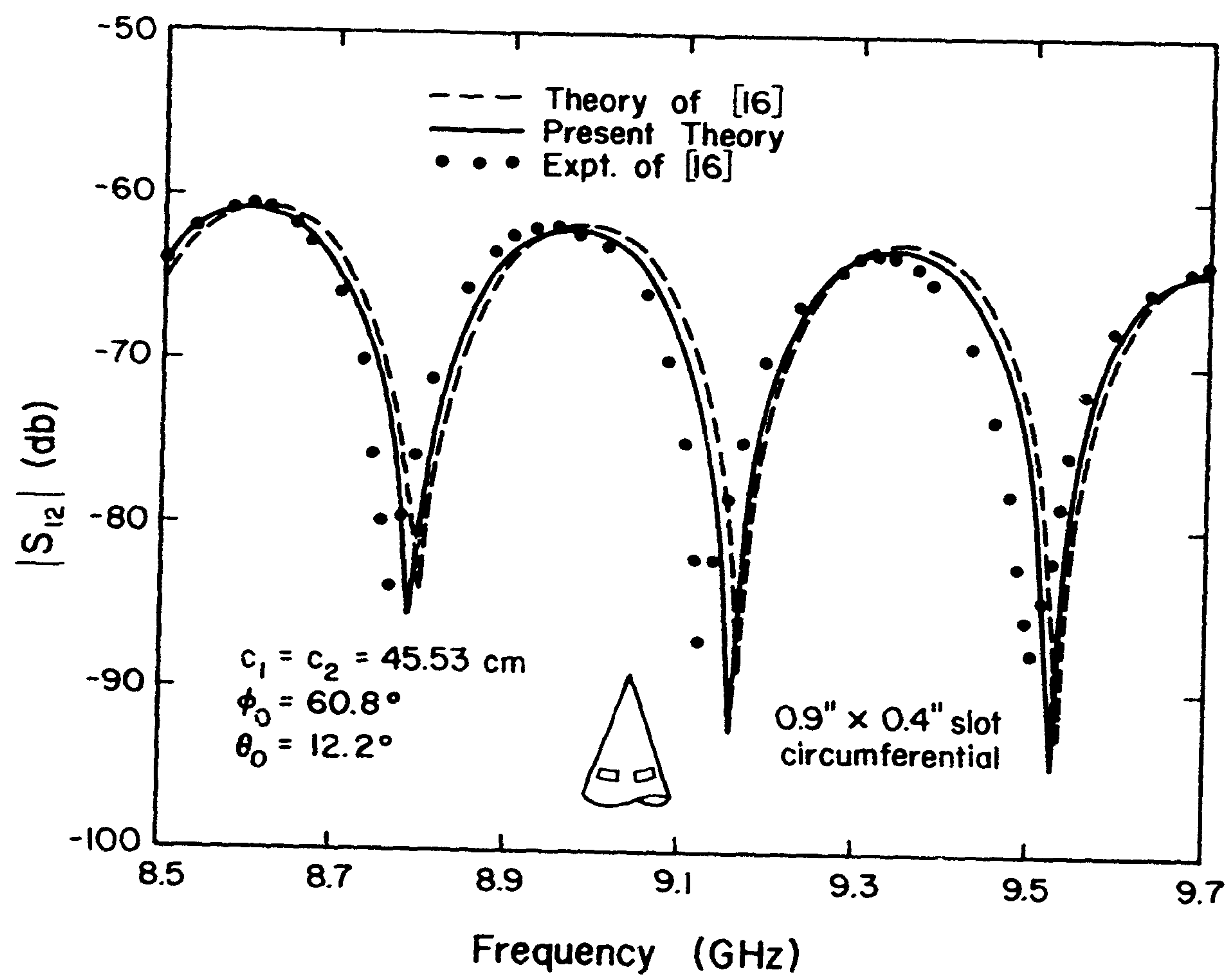


Fig. 6. Coupling coefficient S_{12} between two circumferential slots on a cone as a function of frequency.

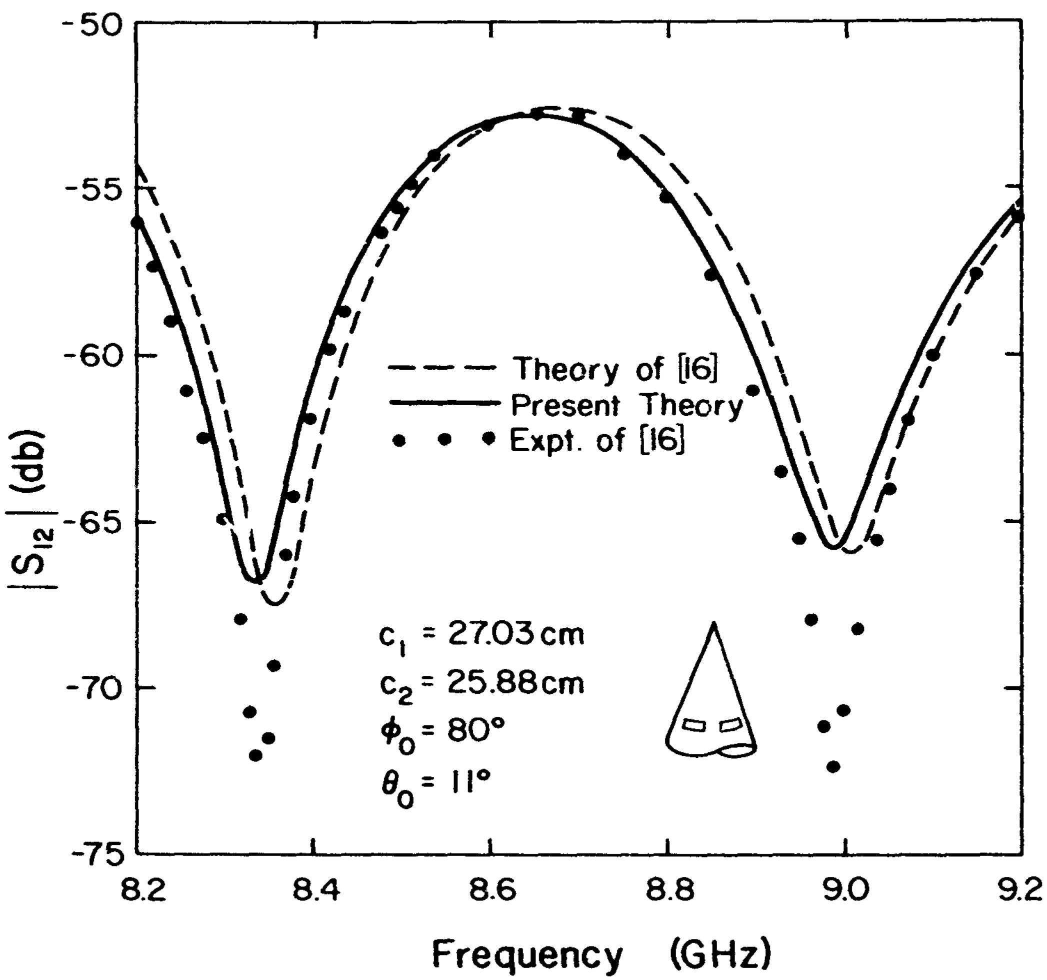


Fig. 7. Coupling coefficient S_{12} between two circumferential slots on a cone as function of frequency.

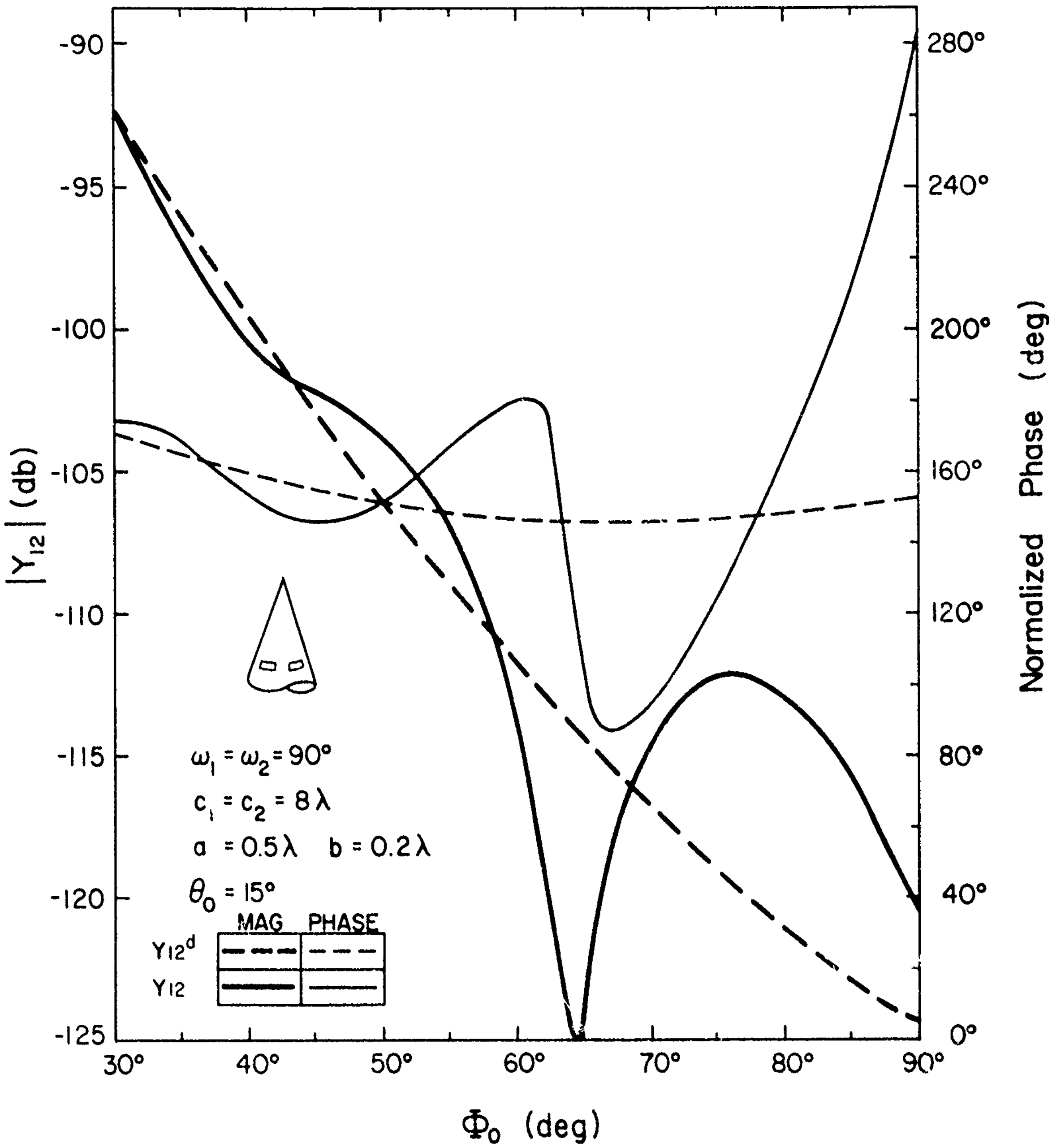


Fig. 8. Mutual admittances on a cone calculated from GTD described in Section 4.

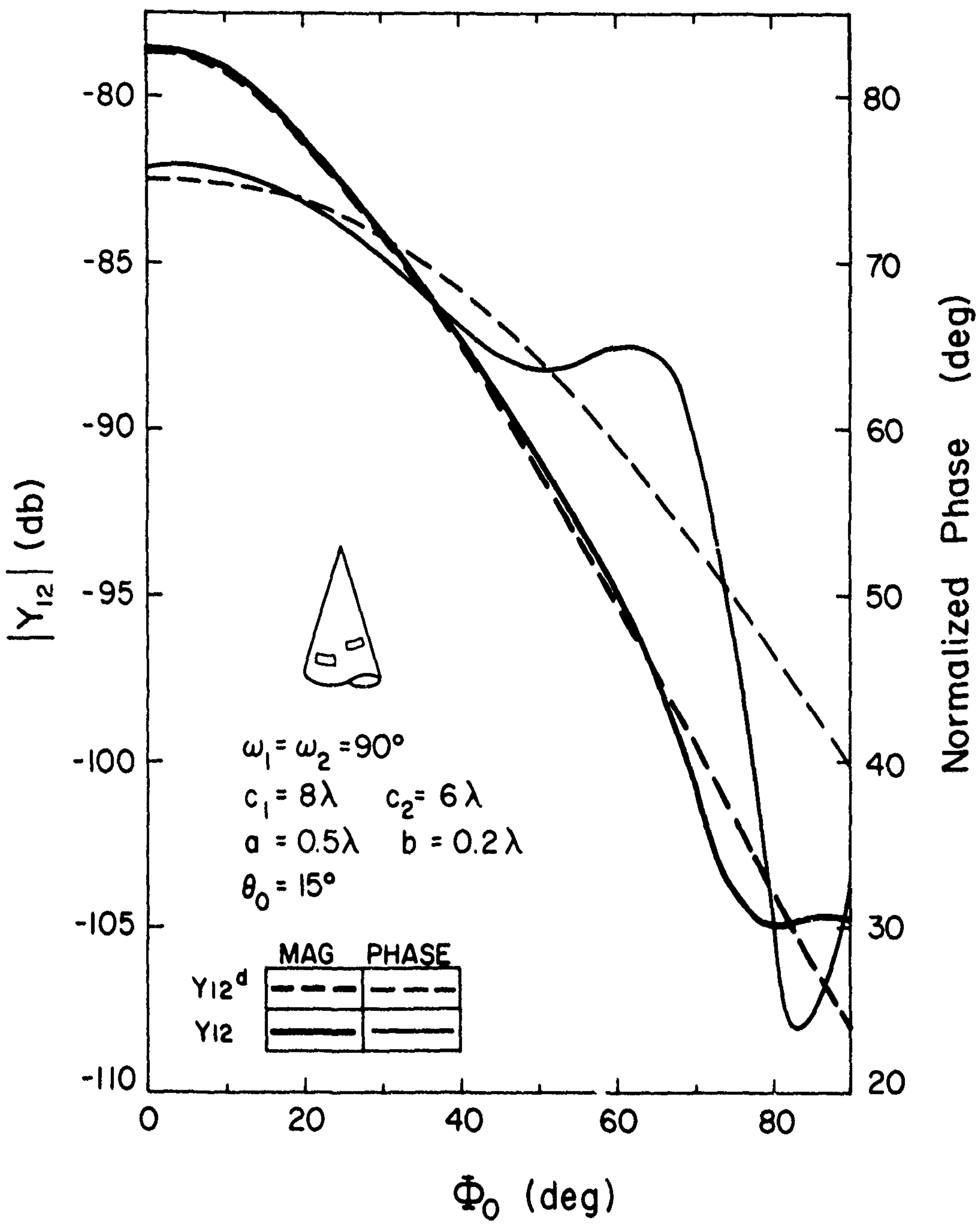
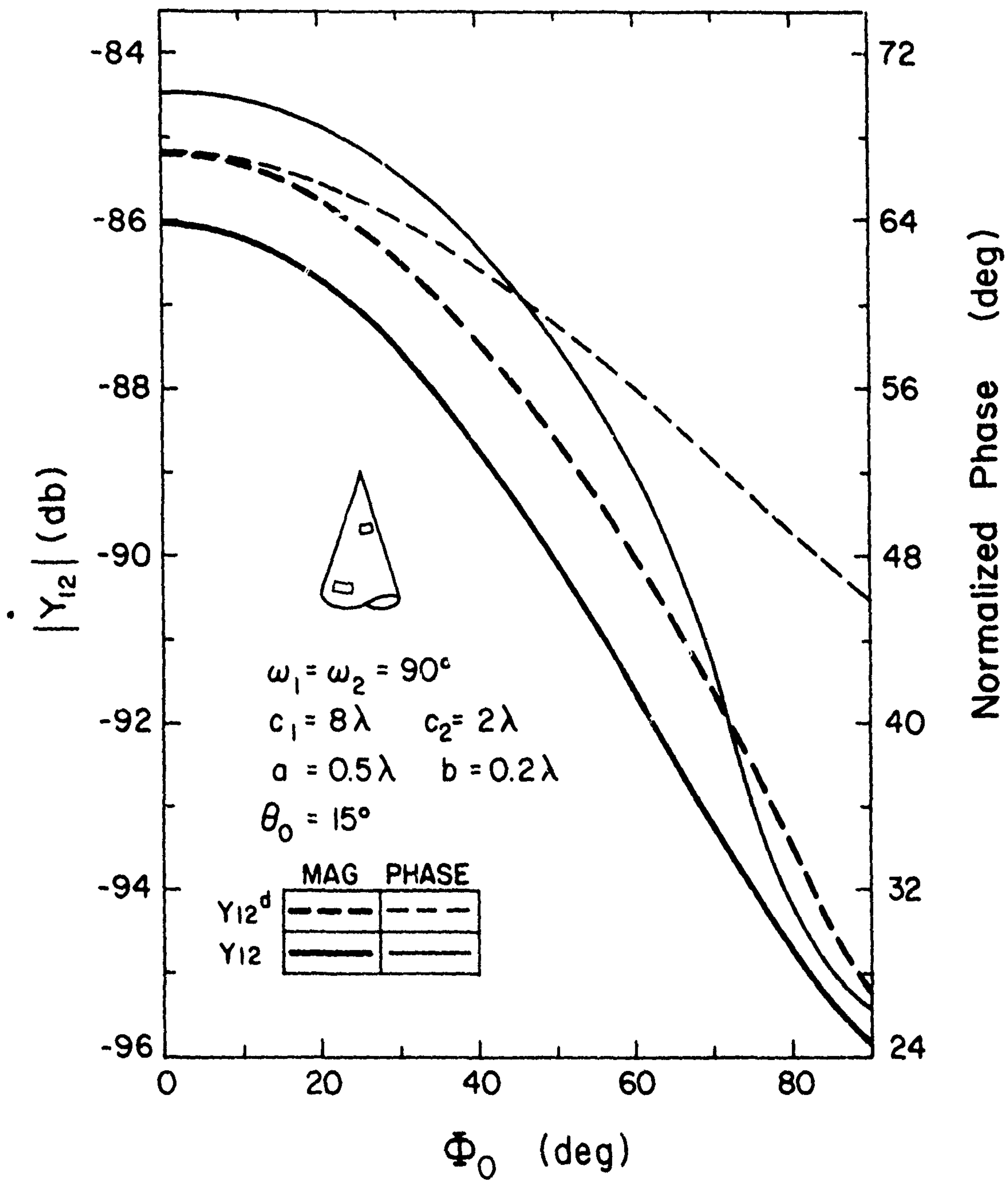


Fig. 9. Mutual admittances on a cone calculated from GTD described in Section 4.



'ig. 10. Mutual admittance on a cone calculated from GTD described in Section 4.

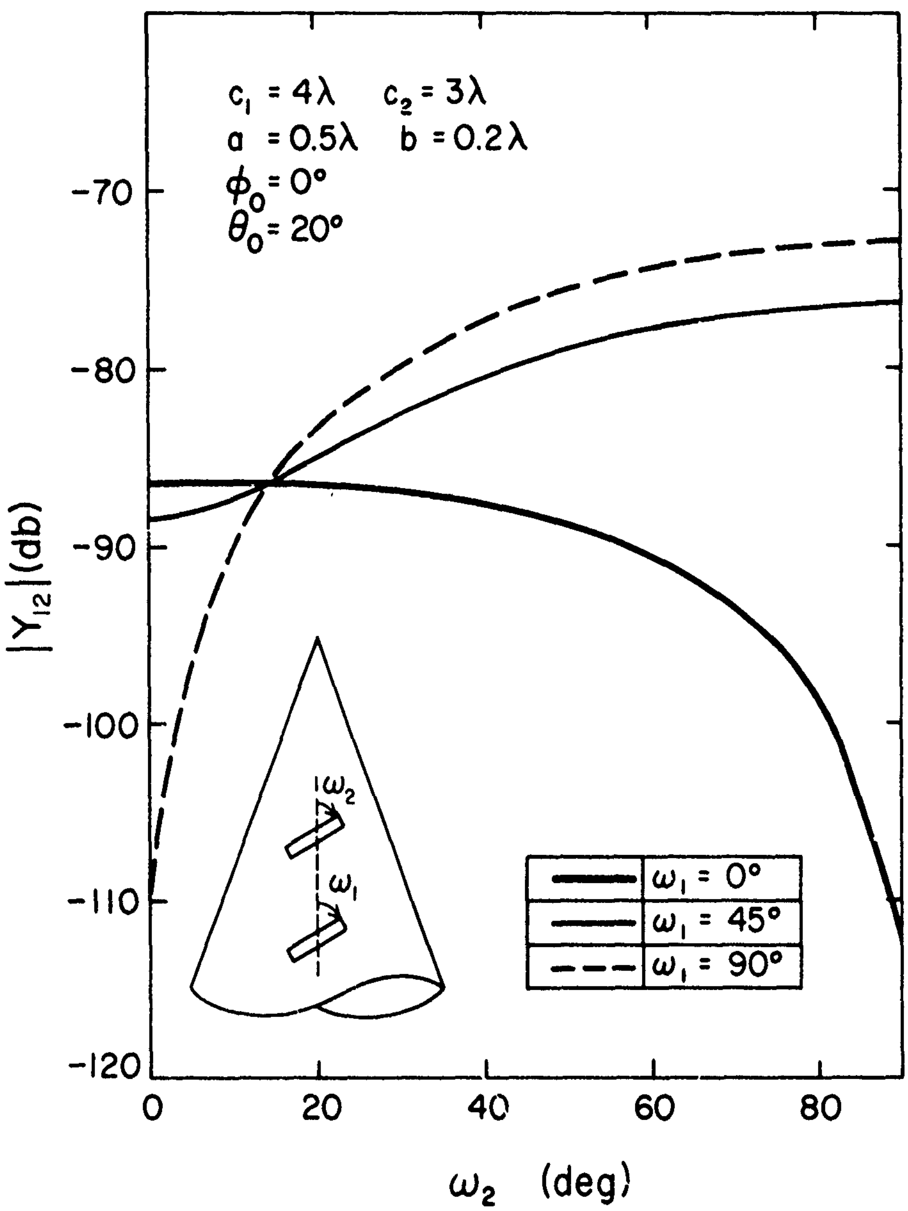


Fig. 11. Mutual admittance Y_{12} between two arbitrarily oriented slots on a cone.

APPENDIX A

FOCK FUNCTIONS

In this appendix we define and list some useful formulas of the functions $w_1(t)$, $w_2(t)$, $v(\xi)$, $u(\xi)$, and $v_1(\xi)$. These functions are commonly known as Fock functions.

(i) Definition: For a complex t and a real ξ,

$$w_1(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} dz \exp\left(tz - \frac{1}{3} z^3\right) \tag{A-1}$$

$$w_2(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_2} dz \exp\left(tz - \frac{1}{3} z^3\right) = w_1^*(t) \tag{A-2}$$

$$v(\xi) = \frac{1}{2} e^{j\pi/4} \xi^{1/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} \frac{w_2(t)}{w_2'(t)} e^{-j\xi t} dt \tag{A-3}$$

$$u(\xi) = e^{j3\pi/4} \xi^{3/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} \frac{w_2'(t)}{w_2(t)} e^{-j\xi t} dt \tag{A-4}$$

$$v_1(\xi) = e^{j3\pi/4} \xi^{3/2} \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} t \frac{w_2(t)}{w_2'(t)} e^{-j\xi t} dt \tag{A-5}$$

where integration contour $\Gamma_1(\Gamma_2)$ goes from ∞ to 0 along the line $\mathrm{Arg}\, z = -2\pi/3$ $(+2\pi/3)$ and from 0 to ∞ along the real axis. Because of different time conventions, $w_1(w_2)$ above is equal to $w_2(w_1)$ defined in [14].

(ii) Residue series representation: For real positive ξ,

$$v(\xi) = e^{-j\pi/4}\sqrt{\pi}\ \xi^{1/2} \sum_{n=1}^{\infty} (t_n')^{-1} e^{-j\xi t_n'} \tag{A-6}$$

$$u(\xi) = e^{j\pi/4} 2\sqrt{\pi}\ \xi^{3/2} \sum_{n=1}^{\infty} e^{-j\xi t_n} \tag{A-7}$$

$$v_1(\xi) = e^{j\pi/4} 2\sqrt{\pi}\ \xi^{3/2} \sum_{n=1}^{\infty} e^{-j\xi t_n'} \tag{A-8}$$

$$v'(\xi) = \frac{1}{2} e^{-j\pi/4}\sqrt{\pi}\ \xi^{-1/2} \sum_{n=1}^{\infty} (1 - j2\xi t_n')(t_n')^{-1} e^{-j\xi t_n'} \tag{A-9}$$

$$u'(\xi) = e^{j\pi/4} 3\sqrt{\pi}\ \xi^{1/2} \sum_{n=1}^{\infty} \left(1 - j\,\frac{2}{3}\,\xi t_n\right) e^{-j\xi t_n} \ . \tag{A-10}$$

where $\{t_n\}$ and $\{t_n'\}$ are zeros of $w_2(t)$ and $w_2'(t)$, respectively, and they are tabulated in [12], [14] and p. 478 of [20].

(iii) Small argument asymptotic expansion: For real positive ξ and $\xi \to 0$,

$$v(\xi) \sim 1 - \frac{\sqrt{\pi}}{4} e^{j\pi/4}\xi^{3/2} + \frac{7j}{60}\xi^3 + \frac{7\sqrt{\pi}}{512} e^{-j\pi/4}\xi^{9/2} - 4.141 \times 10^{-3}\xi^6 + \ldots \tag{A-11}$$

$$u(\xi) \sim 1 - \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} + \frac{5j}{12}\xi^3 + \frac{5\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} - 3.701 \times 10^{-2}\xi^6 + \ldots \tag{A-12}$$

$$v_1(\xi) \sim 1 + \frac{\sqrt{\pi}}{2} e^{j\pi/4}\xi^{3/2} - \frac{7j}{12}\xi^3 - \frac{7\sqrt{\pi}}{64} e^{-j\pi/4}\xi^{9/2} + 4.555 \times 10^{-2}\xi^6 + \ldots \tag{A-13}$$

$$v'(\xi) \sim \frac{3\sqrt{\pi}}{8} e^{-j3\pi/4}\xi^{1/2} + \frac{7j}{20}\xi^2 + \frac{63\sqrt{\pi}}{1024} e^{-j\pi/4}\xi^{7/2} - 2.485 \times 10^{-2}\xi^5 + \ldots \tag{A-14}$$

$$u'(\xi) \sim \frac{3}{4}\sqrt{\pi}\ e^{-j3\pi/4}\xi^{1/2} + \frac{5j}{4}\xi^2 + \frac{45\sqrt{\pi}}{128} e^{-j\pi/4}\xi^{7/2} - 2.221 \times 10^{-1}\xi^5 + \ldots \tag{A-15}$$

(iv) Numerical evaluation: For $\xi \geq \xi_0$, the residue series representation with the first ten terms in the summation may be used. For $\xi \leq \xi_0$, the small argument asymptotic expansion with the first five terms may be used. It has been indicated in [11] that the smoothest crossover is obtained if $\xi_0 = 0.6$. In the present study, we set $\xi_0 = 0.7$, where the difference in the two representations is less than 0.1% in magnitude and 0.9° in phase.

APPENDIX B

COMPUTER LISTING FOR CALCULATING MUTUAL ADMITTANCE ON A CONE BASED ON RAY TECHNIQUE

```
      PROGRAM
     + SCONE (INPUT,TAPE6,OUTPUT,TAPE5=INPUT)
C
C
C
C     ***************************************************
C     *                                                 *
C     *      MUTUAL ADMITTANCE OF SLOTS ON A CONE       *
C     *      ==================================         *
C     *                                                 *
C     *                                                 *
C     *                                                 *
C     *           BY:   S. W.  LEE                      *
C     *                 C. L.  LAW                      *
C     *                 P. CHANG                        *
C     *                                                 *
C     *                                                 *
C     *                                                 *
C     *         DATE:   10/18/77                        *
C     *                                                 *
C     *                                                 *
C     *         UNIVERSITY OF ILLINOIS                  *
C     *                                                 *
C     *                                                 *
C     ***************************************************
C
C
C
C##### PRECISION : SINGLE
C##### LANGUAGE  : FORTRAN
C##### MACHINE   : CDC CYBER 170 SERIES COMPUTER SYSTEM.
C
C
C##### SUBROUTINE NEEDED : FOCK.
C
C##### INPUT FORMAT : FREE.
C
C
C      LATEST REVISIN :  10/18/77
C
C
      IMPLICIT  COMPLEX (C,H,Z) ,REAL(A-B,D-G,K,O-Y)
      REAL ANGLE(20),R11(20),R22(20)
      DIMENSION TITLE(13)
      REAL TN(10),TNPI(10)
      REAL W(7)
      COMMON /DATA1/ TN,TNPI,RHO,C1,C2,F2,IOP,CC,RADN,DEG
      COMMON /CF/ CVF,CUF,CVIF,CVPF,CUPF
      DATA TN/2.33811,4.087,5,5.52156,6.78671,7.99417,
     +            9.02265,10.04017,11.00852,11.93602,12.82878/
      DATA TITLE/13("          ")/
      DATA TNPI/1.01879,3.24820,4.82010,6.16331,7.37218,
```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```
      $              8.48849,9.53545,10.52766,11.47506,12.38479/
C
C
      PI=4.*ATAN(1.E0)
      PI02=2.*PI    $     SQRT2=SQRT(2.)
      DEG=180./PI    $     ACON1=2./3.
      RADN=PI/180.
      PRINT*," INPUT A LINE OF MESSAGE ABOUT THE CURRENT JOB"
      READ(5,911)TITLE
911   FORMAT(13A10)
      PRINT*," HALF CONE ANGLE=",
      READ*,THATHE
      PRINT*," A1,B1,A2,B2=",
      READ*,A1,B1,A2,B2
      PRINT*," W11,W22=",
      READ*,W11,W22
      PRINT*," # OF PHI=",
      READ*,NP
      PRINT*," INPUT PHI(N), ONE ON EACH LINE"
      DO 913 INP=1,NP
913   READ*,ANGLE(INP)
      PRINT*," # OF SETS OF C",
      READ*,NSC
      PRINT*," INPUT C, ONE SET ON EACH LINE"
      DO 916 INSC=1,NSC
916   READ*,R11(INSC),R22(INSC)
      PRINT*," INPUT THE INTEGRATION GRIDS"
      PRINT*," WHICH ARE CORRESPONDED TO (A1,B1,A2,B2)"
      READ*,IP2,IP1,IP4,IP3
      PRINT*," THANK YOU FOR YOUR ACCURATE INPUT!!!"
      WRITE(6,515)TITLE
      WRITE(6,111)
111   FORMAT(" ",14X,46("*")           /,15X,"*",44X,"*"/,15X
     $,"*      MUTUAL ADMITTANCE OF SLOTS ON A CONE    *"/,15X,"*",44X,"*"
     $/,15X,46("*")////)
      WRITE(6,1111)
1111  FORMAT(/,20("*")/,"*"/,"*  FREQUENCY :   K=0.6238D 01 <1/WAVELENGT
     $H>"/"*"/,20("*"))
C
CCCCCCCCCCCCCC
C
C
      THETHA=THATHE*RADN
      SN=SIN(THETHA)
      ATN=TAN(THETHA)
      ACO=COS(THETHA)
      WRITE(6,222)THATHE
222   FORMAT(/,20("*") /,"*"/,"*  GEOMETRY :   HALF CONE ANGLE=",F7.2,
     $" <DEG>"/,"*"/20("*"))
515   FORMAT("1",10X,8A10/10X,8A10///)
      W1=W11*RADN
      W2=W22*RADN
```

```
      WRITE(6,333)A1,B1,A2,B2
333   FORMAT(/,20("*")/,"*"/,"*  SLOT DIMENSION <WAVELENGTH> :   A1=",
     $F7.3,5X,"B1=",F7.3,"  ;  ",5X,"A2=",F7.3,5X,"B2=",F7.3/,"*"/,
     $20("*"))
      WRITE(6,444)W11,W22
444   FORMAT(/,20("*")/,"*"/,"*  SLOT ORIENTATION :   W1=",F7.2,
     $" <DEG>",5X,"W2= ",F7.2," <DEG>"/,"*"/,20("*"))
      WRITE(6,555)IP2,IP1
555   FORMAT(/,20("*")/,"*"/,"*  INTERGRATION GRID :  ",I2," X ",I2/
     $,"*"/,20("*"))
      C1=CEXP(CMPLX(0.E0,-PI/3.))
      C2=CEXP(CMPLX(0.E0, PI/4.))
      CC=C2**3
      ANSR=1.3057/THETHA-1.755+2.772*THETHA-1.459*THETHA**2
      ANSL=2.7195+1.4608*THETHA-1.1295*THETHA**2+0.6566*THETHA**3
      K=2.*PI
      KA1=K*A1
      KB1=K*B1
      KA2=K*A2
      KB2=K*B2
      E2=SQRT(PI)
      WIDTH1=KB2/IP1
      WIDTH2=KA2/IP2
      WIDTH3=KB1/IP3
      WIDTH4=KA1/IP4
      XL1=-KB2/2.-WIDTH1/2.
      XL2=-KA2/2.-WIDTH2/2.
      XL3=-KB1/2.-WIDTH3/2.
      XL4=-KA1/2.-WIDTH4/2.
      DO 100 II=1,NSC
      WRITE(6,666)
666   FORMAT(////,10X,"$$$$$ DATA  OUTPUT  $$$$$")
      KR1=K*R11(II)
      KR2=K*R22(II)
      TERM1=KA1*((KR1+KB1/2.)**2-(KR1-KB1/2.)**2)/(2.*KR1)
      TERM2=KA2*((KR2+KB2/2.)**2-(KR2-KB2/2)**2)/(2.*KR2)
      PART1=KA1*KB1*KA2*KB2*SQRT(ATN/(PIO2*TERM1*TERM2))
      PART2=1./(30.*PI**4*KR1*KR2*SN)
      PART3=SIN(KB1/2.)*SIN(KB2/2.)/(KB1*KB2/4.)
      DDD1=ANSR*COS(ANSL)
      DDD2=ANSR*SIN(ANSL)
      ZYTIP=CMPLX(DDD1,DDD2)
     $      *CEXP(CMPLX(0.E0,PI/4.-KR1-KR2))*PART1*PART2*PART3
     $  *SIN(W1)*SIN(W2)
      TMAG=CABS(ZYTIP)
      IF(TMAG.EQ.0.)GOTO 37
      TDB=20.*ALOG10(TMAG)
      TPHASE=ATAN2(AIMAG(ZYTIP),REAL(ZYTIP))*DEG
37    CONTINUE
      DO 100 I=1,NP
777   FORMAT(//," PHI=",F7.2," <DEG> ;",3X,"C1=",E12.7," <WAVELENGTH>;"
     $,2X,"C2=",E12.7," <WAVELENGTH> ;",3X,"SLOT DIST=",E12.7,
```

```
     $" <WAVELENGTH>"//)
      PHI=ANGLE(I)*RADN
      IF(ABS(PHI).LT..01)PHI=.01
      ZSUM=0.
      DO 10 N1=1,IP4
      TKY1=XL4+WIDTH4*N1
      DO 20 N2=1,IP3
      TKZ1=XL3+WIDTH3*N2
      W5=ATAN2(TKZ1,TKY1)
      IF(W5)1000,2000,2000
1000  W5=W5+PIO2
2000  CONTINUE
      TKY1Z1=TKY1**2+TKZ1**2
      TKR1=ABS(SQRT(KR1**2+TKY1Z1-2.*KR1*SQRT(TKY1Z1)*COS(W1-W5)))
      TPHI1=1./SN*ASIN(SQRT(TKY1Z1)/TKR1*SIN(W1-W5))
      DO 30 N3=1,IP2
      TKY2=XL2+WIDTH2*N3
      DO 40 N4=1,IP1
      TKZ2=XL1+WIDTH1*N4
      W6=ATAN2(TKZ2,TKY2)
      IF(W6)3000,4000,4000
3000  W6=W6+PIO2
4000  CONTINUE
      TKY2Z2=TKY2**2+TKZ2**2
      TKR2=ABS(SQRT(KR2**2+TKY2Z2-2.*KR2*SQRT(TKY2Z2)*COS(W2-W6)))
      TPHI2=PHI+1./SN*ASIN(SQRT(TKY2Z2)/TKR2*SIN(W2-W6))
      KS=SQRT(TKR1**2+TKR2**2-2.*TKR1*TKR2*COS((TPHI2-TPHI1)
     $   *SN))
      OMEGA1=ASIN(TKR2*SIN((TPHI2-TPHI1)*SN)/KS)
      IF(TKR2**2.GT.TKR1**2+KS**2) OMEGA1=PI-OMEGA1
      OMEGA2=OMEGA1+(TPHI2-TPHI1)*SN
      OM1SIN=SIN(OMEGA1)
      OM2SIN=SIN(OMEGA2)
      OM1COS=COS(OMEGA1)
      OM2COS=COS(OMEGA2)
      OM12S=ABS(OM1SIN*OM2SIN)
      OM12C=ABS(OM1COS*OM2COS)
      IF(OM12C.LT.3.E-6)OM12C=3.E-6
      IF(OM12S.LT.3.E-6)OM12S=3.E-6
      UPPER=SQRT(TKR1*TKR2)*ATN
      KRT2=UPPER/OM12S
      KRR2=UPPER/OM12C
      KA=ABS(TKR1*OM1SIN*SN*ACO**2/2.)
     $    **(1./3.)*ABS(TPHI2-TPHI1)
      TAU=((OM1SIN*OM2SIN/ATN)**2/(2.*TKR1*TKR2))
     $     **(1./3.)*KS/KA
      IF(KA.LT.0.7) GO TO 1
      CALL FOCK(KA)
      GO TO 2
1     CALL FOCK1(KA)
2     ZG=(0.,-1.)*CEXP(CMPLX(0.E0,-KS))/(240.*KS*PI**2)
      SQTAU=SQRT(TAU)    $     TAU15=TAU**1.5
```

```
      HB=ZG*(CMPLX(1.E0,-1./KS)*SQTAU*CVF-CUF/KS**2*TAU15
     $    +(0.,1.)*(1./(SQRT2*KRT2))**ACON1*SQTAU*CVFF
     $    +(0.,1.)/(SQRT2*KRT2)**ACON1*KRT2/KRB2*TAU15*CUFF)
      HT=ZG*(0.,1.)/KS*(SQTAU*CVF+CMPLX(1.E0,-2./KS)*TAU15
     $    *CUF+(0.,1.)/(SQRT2*KRT2)**ACON1*TAU15
     $    *CUFF)
      W3=OMEGA1-TPHI1*SN+PI/2.-W1
      W4=OMEGA2-(TPHI2-PHI)*SN+PI/2.-W2
      ZGREEN=COS(W3)*COS(W4)*HB+SIN(W3)*SIN(W4)*HT
      ZSUM=ZSUM+COS(PI/KA1*TKY1)*COS(PI/KA2*TKY2)*ZGREEN
   40 CONTINUE
   30 CONTINUE
   20 CONTINUE
   10 CONTINUE
      ZY12D=ZSUM*WIDTH1*WIDTH2*WIDTH3*WIDTH4
     $          *(-2.)/SQRT(KA1*KB1*KA2*KB2)
      XMAG=CABS(ZY12D)
      PHASE=ATAN2(AIMAG(ZY12D),REAL(ZY12D))*DEG
      DB=20*ALOG10(XMAG)
      DIST=K*SQRT(R11(II)**2+R22(II)**2-2*R11(II)*R22(II)
     $*COS(PHI*SN))
      SDIST=DIST/K
      WRITE(6,777)ANGLE(I),R11(II),R22(II),SDIST
      ZZZZP=ZY12D*CEXP(CMPLX(0.E0,DIST))
      PHASEN=ATAN2(AIMAG(ZZZZP),REAL(ZZZZP))
      PHASEN=PHASEN*DEG
      WRITE(6,888)XMAG,PHASE,DB,PHASEN
  888 FORMAT(2X,'Y12D=',E13.4,' <MHO>',F7.2, ' <DEG>;',5X,'DB= ',E12.5,
     1      ';       NORM PHASE= ',F7.2, ' <DEG>'/)
      WRITE(6,999)TMAG,TPHASE,TDB
  999 FORMAT(2X,'Y12T=',E13.4,' <MHO>',F7.2, ' <DEG>;',5X,'DB= ',E12.5/)
      ZY12=ZY12D+ZYTIP
      XMAG=CABS(ZY12)
      PHASE=ATAN2(AIMAG(ZY12),REAL(ZY12))*DEG
      DB=20.*ALOG10(XMAG)
      ZZZZP=ZY12*CEXP(CMPLX(0.E0,DIST))
      PHASEN=ATAN2(AIMAG(ZZZZP),REAL(ZZZZP))
      PHASEN=PHASEN*DEG
      WRITE(6,889)XMAG,PHASE,DB,PHASEN
  889 FORMAT(2X,'Y12 =',E13.4,' <MHO>',F7.2, ' <DEG>;',5X,'DB= ',E12.5,
     1      ';       NORM PHASE= ',F7.2, ' <DEG>'/)
  100 CONTINUE
      WRITE(6,1919)DDD1,DDD2
 1919 FORMAT(5X,////5X,'REAL=',G10.2,'IMAG=',G10.2)
      STOP
      END
```

```
C
C  THIS SUBROUTINE IS USED TO CALCULATE THE FUNCTIONS CVF,CUF,CV1F,CVPF,CUPF
C
      SUBROUTINE FOCK(X)
      IMPLICIT REAL(A-B,D-H,P-Y),COMPLEX (C,Z)
      REAL TN(10),TNPI(10)
      COMMON/CF/CVF,CUF,CV1F,CVPF,CUPF
      COMMON/DATA1/TN,TNPI,RHO,C1,C2,F2,IOP,CC,RADN,DEG
      F1=SQRT(X)
      F3=X**1.5
      CVF=0.
      CUF=0.
      CV1F=0.
      CVPF=0.
      CUPF=0.
      DO 20 N=1,10
      ZTN=TN(N)*C1
      ZTNPI=TNPI(N)*C1
      C3=CEXP(CMPLX(0.0,-X)*ZTNPI)
      C4=CEXP(CMPLX(0.0,-X)*ZTN)
      CVF=CVF+C3/ZTNPI
      CUF=CUF+C4
      CV1F=CV1F+C3
      CVPF=(1.0-CMPLX(0.0,2*X)*ZTNPI)*C3/ZTNPI+CVPF
      CUPF=(1.0-CMPLX(0.0,2.*X/3.)*ZTN)*C4+CUPF
   20 CONTINUE
      CVF=F2*F1*CVF/C2
      CUF=2.*F2*F3*C2*CUF
      CV1F=2.*F2*F3*C2*CV1F
      CVPF=F2*CVPF/(2.*F1*C2)
      CUPF=3.*F2*F1*C2*CUPF
      RETURN
      ENTRY FOCK1
      XTHREE=X**3
      F1=SQRT(X)
      F3=XTHREE
      Z1=F2*C2*SQRT(F3)
      Z2=CMPLX(0.0,1.0/60.)*XTHREE
      Z3=F2*X**4.5/(C2*64.)
      F4=F3**2
      CVF=1.0-Z1/4.+7*Z2+7.*Z3/8.-4.141E-3*F4
      CUF=1.0-Z1/2.+25.*Z2+5.*Z3-3.701E-2*F4
      CV1F=1.0+Z1/2.-35.*Z2-7.*Z3+4.555E-2*F4
      CVPF=.375*F1*F2/CC+21.*Z2/X+63.*Z3/(16.*X)-2.485E-2*F4/X
      CUPF=.75*F1*F2/CC+CMPLX(0.0,1.25*X**2)+22.5*Z3/X
     1  -2.221E-1*F4/X
      RETURN
      END
```